

Chiral Equivariant Cohomology III

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Dedicated to the memory of our friend and colleague Jerome P. Levine

ABSTRACT. This is the third of a series of papers on a new equivariant cohomology that takes values in a vertex algebra, and contains and generalizes the classical equivariant cohomology of a manifold with a Lie group action à la H. Cartan. In this paper, we compute this cohomology for spheres and show that for any simple connected group G , there is a sphere with infinitely many actions of G which have distinct chiral equivariant cohomology, but identical classical equivariant cohomology. Unlike the classical case, the description of the chiral equivariant cohomology of spheres requires a substantial amount of new structural theory, which we fully develop in this paper. This includes a quasi-conformal structure, equivariant homotopy invariance, and the values of this cohomology on homogeneous spaces. These results rely on crucial features of the underlying vertex algebra valued complex that have no classical analogues.

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1. Introduction

Let G be a compact Lie group with complexified Lie algebra \mathfrak{g} . For a topological G -space M , the equivariant cohomology $H_G^*(M)$ is defined to be $H^*((M \times \mathcal{E})/G)$, where \mathcal{E} is any contractible space on which G acts freely. When M is a manifold on which G acts by diffeomorphisms, there is a de Rham model of $H_G^*(M)$ due to H. Cartan [2][3], and developed further by Duflo-Kumar-Vergne [4] and Guillemin-Sternberg [11]. In fact, one can define the equivariant cohomology $H_G^*(A)$ of any G^* -algebra A . Taking A to be the algebra $\Omega(M)$ of differential forms on M gives us the de Rham model of $H_G^*(M)$, and $H_G^*(\Omega(M)) \cong H_G^*(M)$ by an equivariant version of the de Rham theorem.

In [12], the chiral equivariant cohomology $\mathbf{H}_G^*(\mathcal{A})$ of an $O(\mathfrak{sg})$ -algebra \mathcal{A} was introduced as a vertex algebra analogue of the equivariant cohomology of G^* -algebras. Examples of $O(\mathfrak{sg})$ -algebras include the semi-infinite Weil complex $\mathcal{W}(\mathfrak{g})$, which was introduced in [6], and the chiral de Rham complex $\mathcal{Q}(M)$ of a smooth G -manifold M . The chiral de Rham complex was introduced in [15], and has been studied from several different points of view in recent years [14][10][8][1][19]. In [13], the chiral equivariant cohomology functor was extended to the larger categories of $\mathfrak{sg}[t]$ -algebras and $\mathfrak{sg}[t]$ -modules. Our main example of an $\mathfrak{sg}[t]$ -algebra which is *not* an $O(\mathfrak{sg})$ -algebra is the subalgebra $\mathcal{Q}'(M) \subset \mathcal{Q}(M)$ generated by the weight zero subspace. $\mathbf{H}_G^*(\mathcal{Q}(M))$ and $\mathbf{H}_G^*(\mathcal{Q}'(M))$ are both “chiralizations” of $H_G^*(M)$, that is, vertex algebras equipped with weight gradings

$$\mathbf{H}_G^*(\mathcal{Q}(M)) = \bigoplus_{m \geq 0} \mathbf{H}_G^*(\mathcal{Q}(M))[m], \quad \mathbf{H}_G^*(\mathcal{Q}'(M)) = \bigoplus_{m \geq 0} \mathbf{H}_G^*(\mathcal{Q}'(M))[m],$$

such that $\mathbf{H}_G^*(\mathcal{Q}(M))[0] = H_G^*(M) = \mathbf{H}_G^*(\mathcal{Q}'(M))[0]$. In the case $M = pt$, $\mathcal{Q}(M) = \mathcal{Q}'(M) = \mathbf{C}$, and $\mathbf{H}_G^*(\mathbf{C})$ plays the role of $H_G^*(pt) = S(\mathfrak{g}^*)^G$ in the classical theory.

We briefly recall these constructions, following the notation in [12][13]. We will assume that the reader is familiar with the basic notions in vertex algebra theory. For a list of references, see page 102 of [12]. A *differential vertex algebra* (DVA) is a degree graded vertex algebra $\mathcal{A}^* = \bigoplus_{p \in \mathbf{Z}} \mathcal{A}^p$ equipped with a vertex algebra derivation d of degree 1 such that $d^2 = 0$. A DVA will be called *degree-weight graded* if it has an additional $\mathbf{Z}_{\geq 0}$ -grading by weight, which is compatible with the degree in the sense that $\mathcal{A}^p = \bigoplus_{n \geq 0} \mathcal{A}^p[n]$. There is an auxiliary structure on a DVA which is analogous to the structure of a G^* -algebra in [11]. Associated to \mathfrak{g} is a Lie superalgebra $\mathfrak{sg} := \mathfrak{g} \triangleright \mathfrak{g}^{-1}$ with bracket $[(\xi, \eta), (x, y)] = ([\xi, x], [\xi, y] - [x, \eta])$, which is equipped with a differential

$d : (\xi, \eta) \mapsto (\eta, 0)$. This differential extends to the loop algebra $\mathfrak{sg}[t, t^{-1}]$, and gives rise to a vertex algebra derivation on the corresponding current algebra $O(\mathfrak{sg}) := O(\mathfrak{sg}, 0)$. Here 0 denotes the zero bilinear form on \mathfrak{sg} .

An $O(\mathfrak{sg})$ -algebra is a degree-weight graded DVA \mathcal{A} equipped with a DVA homomorphism $\rho : O(\mathfrak{sg}) \rightarrow \mathcal{A}$, which we denote by $(\xi, \eta) \rightarrow L_\xi + \iota_\eta$. Although this definition makes sense for any Lie algebra \mathfrak{g} , we will assume throughout this paper that \mathfrak{g} is the Lie algebra of G , and we require \mathcal{A} to admit an action $\hat{\rho} : G \rightarrow \text{Aut}(\mathcal{A})$ of G by vertex algebra automorphisms which is compatible with the $O(\mathfrak{sg})$ -structure in the following sense:

$$\frac{d}{dt} \hat{\rho}(\exp(t\xi))|_{t=0} = L_\xi(0), \quad (1.1)$$

$$\hat{\rho}(g)L_\xi(n)\hat{\rho}(g^{-1}) = L_{Ad(g)(\xi)}(n), \quad (1.2)$$

$$\hat{\rho}(g)\iota_\xi(n)\hat{\rho}(g^{-1}) = \iota_{Ad(g)(\xi)}(n), \quad (1.3)$$

$$\hat{\rho}(g)d\hat{\rho}(g^{-1}) = d, \quad (1.4)$$

for all $\xi \in \mathfrak{g}$, $g \in G$, and $n \in \mathbf{Z}$. These conditions are analogous to Equations (2.23)-(2.26) of [11]. In order for (1.1) to make sense, we must be able to differentiate along appropriate curves in \mathcal{A} , which is the case in our main example $\mathcal{A} = \mathcal{Q}(M)$.

In [13], we observed that the chiral equivariant cohomology functor can be defined on the larger class of spaces which carry only a representation of the Lie subalgebra $\mathfrak{sg}[t]$ of $\mathfrak{sg}[t, t^{-1}]$.

Definition 1.1. *An $\mathfrak{sg}[t]$ -module is a degree-weight graded complex $(\mathcal{A}, d_{\mathcal{A}})$ equipped with a Lie algebra homomorphism $\rho : \mathfrak{sg}[t] \rightarrow \text{End } \mathcal{A}$, such that for all $x \in \mathfrak{sg}[t]$ we have*

- $\rho(dx) = [d_{\mathcal{A}}, \rho(x)]$
- $\rho(x)$ has degree 0 whenever x is even in $\mathfrak{sg}[t]$, and degree -1 whenever x is odd, and has weight $-n$ if $x \in \mathfrak{sg}t^n$.

As above, we will always assume that \mathfrak{g} is the Lie algebra of G , and \mathcal{A} has an action of G which is compatible with the $\mathfrak{sg}[t]$ -structure, i.e., (1.1)-(1.4) hold for $n \geq 0$.

Definition 1.2. *Given an $\mathfrak{sg}[t]$ -module (\mathcal{A}, d) , we define the chiral horizontal, invariant and basic subspaces of \mathcal{A} to be respectively*

$$\begin{aligned}\mathcal{A}_{hor} &= \{a \in \mathcal{A} | \rho(x)a = 0 \ \forall x \in \mathfrak{g}^{-1}[t]\} \\ \mathcal{A}_{inv} &= \{a \in \mathcal{A} | \rho(x)a = 0 \ \forall x \in \mathfrak{g}[t], \ \hat{\rho}(g)(a) = a \ \forall g \in G\} \\ \mathcal{A}_{bas} &= \mathcal{A}_{hor} \cap \mathcal{A}_{inv}.\end{aligned}$$

An $\mathfrak{sg}[t]$ -module (\mathcal{A}, d) is called an $\mathfrak{sg}[t]$ -algebra if it is also a DVA such that $\mathcal{A}_{hor}, \mathcal{A}_{inv}$ are both vertex subalgebras of \mathcal{A} , and G acts by DVA automorphisms.

When we are working with multiple groups G, H, \dots we will use the notations $\mathcal{A}_{G-hor}, \mathcal{A}_{G-inv}, \mathcal{A}_{G-bas}$, etc., to avoid confusion. Given an $\mathfrak{sg}[t]$ -module (\mathcal{A}, d) , $\mathcal{A}_{inv}, \mathcal{A}_{bas}$ are both subcomplexes of \mathcal{A} , but \mathcal{A}_{hor} is not a subcomplex of \mathcal{A} in general. The Lie algebra $\mathfrak{sg}[t]$ is not required to act by derivations on a DVA \mathcal{A} to make it an $\mathfrak{sg}[t]$ -algebra. If (\mathcal{A}, d) is an $O(\mathfrak{sg})$ -algebra, any subDVA \mathcal{B} which is closed under the operators $(L_\xi + \iota_\eta) \circ_p, p \geq 0$, is an $\mathfrak{sg}[t]$ -algebra.

Definition 1.3. *For any $\mathfrak{sg}[t]$ -module $(\mathcal{A}, d_{\mathcal{A}})$, we define its chiral basic cohomology $\mathbf{H}_{bas}^*(\mathcal{A})$ to be $H^*(\mathcal{A}_{bas}, d_{\mathcal{A}})$. We define its chiral equivariant cohomology $\mathbf{H}_G^*(\mathcal{A})$ to be $\mathbf{H}_{bas}^*(\mathcal{W}(\mathfrak{g}) \otimes \mathcal{A})$. The differential on $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{A}$ is $d_{\mathcal{W}} \otimes 1 + 1 \otimes d_{\mathcal{A}}$, where $d_{\mathcal{W}} = J(0) + K(0)$, as in [12].*

In this paper, our main focus is on the cases $\mathcal{A} = \mathcal{Q}(M)$ and $\mathcal{A} = \mathcal{Q}'(M)$, for a smooth G -manifold M . Recall from [13] that for each $m \geq 0$, $\mathcal{Q}_M[m]$ is a sheaf of vector spaces on M , and $\mathcal{Q}(M)$ is the space of global sections of the weak sheaf of vertex algebras $\mathcal{Q}_M = \oplus_{m \geq 0} \mathcal{Q}_M[m]$ on M . Similarly, $\mathcal{Q}'_M[m]$ is the subsheaf of $\mathcal{Q}_M[m]$ generated by the weight zero subspace, and $\mathcal{Q}'(M)$ is the space of global sections of the weak sheaf of abelian vertex algebras $\mathcal{Q}'_M = \oplus_{m \geq 0} \mathcal{Q}'_M[m]$. In this terminology, a *weak sheaf* is a presheaf which satisfies a slightly weaker version of the reconstruction axiom:

$$0 \rightarrow \mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \quad (1.5)$$

is exact for *finite* open covers $\{U_i\}$ of an open set U (see Section 1.1 of [13]). Whenever we need to construct a global section of \mathcal{Q}_M or \mathcal{Q}'_M by gluing together local sections, these

sections are always homogeneous of finite weight, so we may work inside the sheaf $\mathcal{Q}_M[m]$ or $\mathcal{Q}'_M[m]$ for some m .

In [12][13] we proved a number of structural results about the chiral equivariant cohomology, which we recall below.

- The functor $\mathbf{H}_G^*(\mathcal{Q}'(-))$ is contravariant in M for any G . For fixed M , $\mathbf{H}_{(-)}^*(\mathcal{Q}'(M))$ is not functorial in G in general, but is contravariant with respect to abelian groups.
- For any group G of positive dimension, $\mathbf{H}_G^*(\mathbf{C})$ contains nonzero classes in every positive weight. If G is semisimple, $\mathbf{H}_G^*(\mathbf{C})$ is a conformal vertex algebra with Virasoro element \mathbf{L} of central charge zero.
- For any $\mathfrak{sg}[t]$ -algebra \mathcal{A} , the canonical map

$$\kappa_G : \mathbf{H}_{bas}^*(\mathcal{W}(\mathfrak{g})) = \mathbf{H}_G^*(\mathbf{C}) \rightarrow \mathbf{H}_G^*(\mathcal{A})$$

induced by $\mathcal{W}(\mathfrak{g}) \hookrightarrow \mathcal{W}(\mathfrak{g}) \otimes \mathcal{A}$ is called the *chiral Chern-Weil map* of \mathcal{A} . For $\mathcal{A} = \mathcal{Q}(M)$ or $\mathcal{A} = \mathcal{Q}'(M)$, this map extends the classical Chern-Weil map $H_G^*(pt) \rightarrow H_G^*(M)$.

- If M^G is nonempty, $\kappa_G : \mathbf{H}_G^*(\mathbf{C}) \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(M))$ is injective. If G is semisimple, $\mathbf{H}_G^*(\mathcal{Q}'(M))$ is then a nontrivial conformal vertex algebra with Virasoro element $\kappa_G(\mathbf{L})$ of central charge zero.
- If the action of G on M is locally free, $\mathbf{H}_G^*(\mathcal{Q}'(M))_+ = 0$ and $\mathbf{H}_G^*(\mathcal{Q}(M))_+ = 0$.
- If G is a torus T , $\mathbf{H}_T^*(\mathcal{Q}'(M))_+ = 0$ if and only if the action of T is locally free. The converse fails in general. For example, if G is simple and $T \subset G$ is a torus, $\mathbf{H}_G^*(\mathcal{Q}'(G/T))_+ = 0$ even though the action of G on G/T is not locally free.
- If G is semisimple and V is a faithful linear representation of G , $\mathbf{H}_G^*(\mathcal{Q}(V))_+ = 0$.

1.1. Outline of main results

In this paper, we continue the study of $\mathbf{H}_G^*(\mathcal{Q}(M))$ and $\mathbf{H}_G^*(\mathcal{Q}'(M))$. Our goal is to understand what kind of geometric information is contained in the positive-weight subspaces

$$\mathbf{H}_G^*(\mathcal{Q}(M))_+ = \bigoplus_{m>0} \mathbf{H}_G^*(\mathcal{Q}(M))[m], \quad \mathbf{H}_G^*(\mathcal{Q}'(M))_+ = \bigoplus_{m>0} \mathbf{H}_G^*(\mathcal{Q}'(M))[m].$$

There are three basic results we need to establish. First, for G -invariant open sets $U, V \subset M$, there exist Mayer-Vietoris sequences

$$\begin{aligned} \cdots \rightarrow \mathbf{H}_G^p(\mathcal{Q}(U \cup V)) &\rightarrow \mathbf{H}_G^p(\mathcal{Q}(U)) \oplus \mathbf{H}_G^p(\mathcal{Q}(V)) \rightarrow \mathbf{H}_G^p(\mathcal{Q}(U \cap V)) \rightarrow \cdots, \\ \cdots \rightarrow \mathbf{H}_G^p(\mathcal{Q}'(U \cup V)) &\rightarrow \mathbf{H}_G^p(\mathcal{Q}'(U)) \oplus \mathbf{H}_G^p(\mathcal{Q}'(V)) \rightarrow \mathbf{H}_G^p(\mathcal{Q}'(U \cap V)) \rightarrow \cdots. \end{aligned}$$

Second, $\mathbf{H}_G^*(\mathcal{Q}'(-))$ is invariant under G -equivariant homotopy. That is, if M and N are G -manifolds and $\phi_0, \phi_1 : M \rightarrow N$ are equivariantly homotopic G -maps, the induced maps $\phi_0^*, \phi_1^* : \mathbf{H}_G^*(\mathcal{Q}'(N)) \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(M))$ are the same.

Third, for any G and M , $\mathbf{H}_G^*(\mathcal{Q}'(M))$ and $\mathbf{H}_G^*(\mathcal{Q}(M))$ have *quasi-conformal structures*. That is, they admit an action of the subalgebra of the Virasoro algebra generated by $\{L_n \mid n \geq -1\}$, such that L_{-1} acts by ∂ and L_0 acts by $n \cdot id$ on the subspace of weight n . The quasi-conformal structure provides a powerful vanishing criterion for $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ and $\mathbf{H}_G^*(\mathcal{Q}(M))_+$; it suffices to show that L_0 acts by zero.

Using these three basic tools, our goal will be to give a *relative* description of $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ and $\mathbf{H}_G^*(\mathcal{Q}(M))_+$ in terms of the vertex algebras $\mathbf{H}_K^*(\mathbf{C})$ for connected normal subgroups K of G , together with geometric data about M . If K is abelian, Theorem 6.1 of [12] gives a complete description of $\mathbf{H}_K^*(\mathbf{C})$, but if K is non-abelian $\mathbf{H}_K^*(\mathbf{C})$ is still a rather mysterious object. Computer calculations in the cases $K = SU(2)$ and $K = SU(3)$ indicate that $\mathbf{H}_K^*(\mathbf{C})$ has a rich structure and contains many elements beyond the Virasoro element \mathbf{L} that have no classical analogues.

Since G -manifolds locally look like vector bundles over homogeneous spaces, a basic problem is to compute $\mathbf{H}_G^*(\mathcal{Q}'(G/H))$ for any closed subgroup $H \subset G$.

Theorem 1.4. *For any compact, connected G and closed subgroup $H \subset G$,*

$$\mathbf{H}_G^*(\mathcal{Q}'(G/H)) \cong \mathbf{H}_{K_0}^*(\mathbf{C}) \otimes H_{G'}^*(G/H), \quad (1.6)$$

where K_0 is the identity component of $K = \text{Ker}(G \rightarrow \text{Diff}(G/H))$, and $G' = G/K_0$. Here $H_{G'}^*(G/H)$ is regarded as a vertex algebra in which all circle products are trivial except \circ_{-1} , and (1.6) is a vertex algebra isomorphism.

Theorem 1.4 generalizes Corollary 6.14 of [13], which deals with the case where G is semisimple and H is a torus. A consequence of this result is that for compact M , the degree p and weight n subspace $\mathbf{H}_G^p(\mathcal{Q}'(M))[n]$ is finite-dimensional for all $p \in \mathbf{Z}$ and $n \geq 0$, which extends a well-known classical result in the case $n = 0$.

Next, we study $\mathbf{H}_G^*(\mathcal{Q}(M))$ via the map $\mathbf{H}_G^*(\mathcal{Q}'(M)) \rightarrow \mathbf{H}_G^*(\mathcal{Q}(M))$ induced by the inclusion $\mathcal{Q}'(M) \hookrightarrow \mathcal{Q}(M)$.

Theorem 1.5. *For any G -manifold M ,*

$$\mathbf{H}_G^*(\mathcal{Q}(M)) \cong \mathbf{H}_{K_0}^*(\mathbf{C}) \otimes H_{G'}^*(M),$$

where K_0 is the identity component of $K = \text{Ker}(G \rightarrow \text{Diff}(M))$ and $G' = G/K_0$. In particular, $\mathbf{H}_G^*(\mathcal{Q}(M))_+ = 0$ whenever K is finite.

Thus $\mathbf{H}_G^*(\mathcal{Q}(M))_+$ depends only on K_0 , so it carries no other geometric information about M . An important consequence is that for any G and M , $\mathbf{H}_G^*(\mathcal{Q}(M))_+$ and $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ are *vertex algebra ideals*, i.e., they are closed under $\alpha \circ_n$ and $\circ_n \alpha$ for all $n \in \mathbf{Z}$ and α in $\mathbf{H}_G^*(\mathcal{Q}(M))$, $\mathbf{H}_G^*(\mathcal{Q}'(M))$, respectively.

Next, we study $\mathbf{H}_G^*(\mathcal{Q}'(M))$, which in contrast to $\mathbf{H}_G^*(\mathcal{Q}(M))$, carries non-trivial geometric information about M beyond weight zero. We focus on three special cases: G simple, $G = G_1 \times G_2$ where G_1, G_2 are simple, and G abelian.

Theorem 1.6. *(Positive-weight localization for simple group actions) For any simple G and G -manifold M , the map*

$$\mathbf{H}_G^*(\mathcal{Q}'(M))_+ \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(M^G))_+ \quad (1.7)$$

induced by the inclusion $i : M^G \rightarrow M$, is an isomorphism of vertex algebra ideals. Hence $\mathbf{H}_G^*(\mathcal{Q}'(M))_+ \cong \mathbf{H}_G^*(\mathbf{C})_+ \otimes H^*(M^G)$. Moreover, both the ring structure of $H^*(M^G)$ and the map $i^* : H_G^*(M) \rightarrow H_G^*(M^G)$ are encoded in the vertex algebra structure of $\mathbf{H}_G^*(\mathcal{Q}'(M))$.

Using Theorem 1.6, together with results of R. Oliver [16][17] which describe the fixed-point subsets of group actions on contractible spaces, we prove our main theorem:

Theorem 1.7. *For any simple G , there is a sphere with infinitely many smooth actions of G , which have pairwise distinct chiral equivariant cohomology, but identical classical equivariant cohomology.*

One can even construct a morphism $f : M \rightarrow N$ in the category of compact G -manifolds which induces a ring isomorphism $H_G^*(N) \rightarrow H_G^*(M)$ (with \mathbf{Z} -coefficients), such that $\mathbf{H}_G^*(\mathcal{Q}'(M)) \neq \mathbf{H}_G^*(\mathcal{Q}'(N))$. Hence $\mathbf{H}_G^*(\mathcal{Q}'(-))$ is a strictly stronger invariant than $H_G^*(-)$ on the category of compact G -manifolds.

Similarly, in the case $G = G_1 \times G_2$ where G_1, G_2 are simple, we describe $\mathbf{H}_G^*(\mathcal{Q}'(M))$ in terms of the vertex algebras $\mathbf{H}_{G_1}^*(\mathbf{C})$, $\mathbf{H}_{G_2}^*(\mathbf{C})$ and the rings $H^*(M^G)$, $H_{G_1}^*(M^{G_2})$, and $H_{G_2}^*(M^{G_1})$.

The case where G is the circle S^1 is analogous to the case of simple G . When G is a general torus T , $\mathbf{H}_T^*(\mathcal{Q}'(M))_+$ will typically depend on the family of rings $H_{T/T'}^*(M^{T'})$ for all subtori $T' \subset T$ for which $M^{T'}$ is non-empty, and can be quite complicated. As an example, we compute $\mathbf{H}_T^*(\mathcal{Q}'(\mathbf{CP}^2))$, where $T = S^1 \times S^1$ and \mathbf{CP}^2 is equipped with the usual linear action.

We conclude with a few remarks about $\mathbf{H}_G^*(\mathbf{C})$. In [6], it was suggested that the semi-infinite Weil complex $\mathcal{W}(\mathfrak{g})$ should play a role in semi-infinite geometry analogous to the role of the classical Weil complex $W(\mathfrak{g})$. Note that $H_G^*(pt) = S(\mathfrak{g}^*)^G$ can be regarded either as the basic cohomology $H_{bas}^*(W(\mathfrak{g}))$, or as the Lie algebra cohomology $H^0(\mathfrak{g}, S(\mathfrak{g}^*))$. The analogue of $H^0(\mathfrak{g}, S(\mathfrak{g}^*))$ is the *semi-infinite cohomology* $H^{\infty+*}(\hat{\mathfrak{g}}, \mathcal{S}(\mathfrak{g}))$ [5][7][20], whereas the analogue of $H_{bas}^*(W(\mathfrak{g}))$ is $\mathbf{H}_{bas}^*(\mathcal{W}(\mathfrak{g})) = \mathbf{H}_G^*(\mathbf{C})$. Here $\mathcal{S}(\mathfrak{g})$ is the semi-infinite symmetric algebra on \mathfrak{g} . In contrast to the classical case, $\mathbf{H}_G^*(\mathbf{C}) \neq H^{\infty+*}(\hat{\mathfrak{g}}, \mathcal{S}(\mathfrak{g}))$. It

would be interesting to construct an equivariant cohomology theory for manifolds in which $H^{\infty+*}(\hat{\mathfrak{g}}, \mathcal{S}(\mathfrak{g}))$ plays the role of $S(\mathfrak{g}^*)^G$ (as suggested in [6]), and compare it to our theory.

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1.2. General remarks about group actions on manifolds

Let G_0 denote the identity component of G . For any G -manifold M , the finite group $\Gamma = G/G_0$ acts on the complex $(\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G_0-bas}$ by DVA automorphisms, and we have

$$(\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G-bas} = ((\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G_0-bas})^G = ((\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G_0-bas})^\Gamma.$$

Since the differential $d_{\mathcal{W}} + d_{\mathcal{Q}}$ commutes with the action of Γ on $(\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G_0-bas}$, Γ acts on $\mathbf{H}_{G_0}^*(\mathcal{Q}(M))$, and we have $\mathbf{H}_G^*(\mathcal{Q}(M)) = \mathbf{H}_{G_0}^*(\mathcal{Q}(M))^\Gamma$. Similarly, $\mathbf{H}_G^*(\mathcal{Q}'(M)) = \mathbf{H}_{G_0}^*(\mathcal{Q}'(M))^\Gamma$. Hence there is essentially no new content in studying $\mathbf{H}_G^*(-)$ for disconnected groups, and for the remainder of this paper, we will only consider the functor $\mathbf{H}_G^*(-)$ for connected G .

We say that G acts *effectively* on M if $K = \text{Ker}(G \rightarrow \text{Diff}(M))$ is trivial, and we say that G acts *almost effectively* if K is finite. Let K_0 denote the identity component of K and let $G' = G/K_0$.

Lemma 1.8. *Let M be a G -manifold, and suppose that $K = \text{Ker}(G \rightarrow \text{Diff}(M))$ has positive dimension. Then G' acts almost effectively on M and*

$$\mathbf{H}_G^*(\mathcal{Q}(M)) = \mathbf{H}_{K_0}^*(\mathbf{C}) \otimes \mathbf{H}_{G'}^*(\mathcal{Q}(M)), \quad \mathbf{H}_G^*(\mathcal{Q}'(M)) = \mathbf{H}_{K_0}^*(\mathbf{C}) \otimes \mathbf{H}_{G'}^*(\mathcal{Q}'(M)).$$

Proof: Clearly $K/K_0 = \text{Ker}(G' \rightarrow \text{Diff}(M))$, which is finite because K is compact. Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{g}'$ where \mathfrak{k} and \mathfrak{g}' are the Lie algebras of K and G' , respectively, we have $\mathcal{W}(\mathfrak{g}) = \mathcal{W}(\mathfrak{k}) \otimes \mathcal{W}(\mathfrak{g}')$. Then

$$(\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(M))_{G-bas} = \mathcal{W}(\mathfrak{k})_{K_0-bas} \otimes (\mathcal{W}(\mathfrak{g}') \otimes \mathcal{Q}'(M))_{G'-bas}.$$

Note that the differential $d = d_{\mathcal{W}(\mathfrak{g})} + d_{\mathcal{Q}}$ of $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M)$ can be written as $d_{\mathcal{W}(\mathfrak{k})} + d_{\mathcal{W}(\mathfrak{g}')} + d_{\mathcal{Q}}$, and these three terms pairwise commute. Since $d_{\mathcal{W}(\mathfrak{k})}$ only acts on $\mathcal{W}(\mathfrak{k})_{K_0-bas}$ and $d_{\mathcal{W}(\mathfrak{g}')} + d_{\mathcal{Q}}$ only acts on $\mathcal{W}(\mathfrak{g}') \otimes \mathcal{Q}(M))_{G'-bas}$, the claim follows. \square

Any compact connected G has a finite cover of the form $\tilde{G} = G_1 \times \cdots \times G_k \times T$, where the G_i are simple and T is a torus. If M is a G -manifold, the action can be lifted to \tilde{G} so that $\Gamma = \text{Ker}(\tilde{G} \rightarrow G) \subset \text{Ker}(\tilde{G} \rightarrow \text{Diff}(M))$.

Lemma 1.9. *Suppose M is a G -manifold, $K = \text{Ker}(G \rightarrow \text{Diff}(M))$, and Γ is a finite subgroup of K . Then G/Γ acts on M and*

$$\mathbf{H}_G^*(\mathcal{Q}'(M)) = \mathbf{H}_{G/\Gamma}^*(\mathcal{Q}'(M)), \quad \mathbf{H}_G^*(\mathcal{Q}(M)) = \mathbf{H}_{G/\Gamma}^*(\mathcal{Q}(M)).$$

In particular, if G , \tilde{G} , and Γ are as above, we have

$$\mathbf{H}_G^*(\mathcal{Q}'(M)) = \mathbf{H}_{\tilde{G}}^*(\mathcal{Q}'(M)), \quad \mathbf{H}_G^*(\mathcal{Q}(M)) = \mathbf{H}_{\tilde{G}}^*(\mathcal{Q}(M)).$$

Proof: Γ acts trivially on $\mathcal{Q}(M)$ since it acts trivially on M . The adjoint and coadjoint actions of Γ on \mathfrak{g} and \mathfrak{g}^* are trivial, so Γ also acts trivially on $\mathcal{W}(\mathfrak{g})$. Thus the G -invariance and the G/Γ -invariance conditions on $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(M)$ and $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M)$ are the same. Since G and G/Γ have the same Lie algebra, the $\mathfrak{sg}[t]$ -basic condition is also the same. \square

2. Mayer-Vietoris Sequences

In this section, we show that for G -invariant open sets $U, V \subset M$, there exist Mayer-Vietoris sequences

$$\begin{aligned} \cdots \rightarrow \mathbf{H}_G^p(\mathcal{Q}'(U \cup V)) &\rightarrow \mathbf{H}_G^p(\mathcal{Q}'(U)) \oplus \mathbf{H}_G^p(\mathcal{Q}'(V)) \rightarrow \mathbf{H}_G^p(\mathcal{Q}'(U \cap V)) \rightarrow \cdots, \\ \cdots \rightarrow \mathbf{H}_G^p(\mathcal{Q}(U \cup V)) &\rightarrow \mathbf{H}_G^p(\mathcal{Q}(U)) \oplus \mathbf{H}_G^p(\mathcal{Q}(V)) \rightarrow \mathbf{H}_G^p(\mathcal{Q}(U \cap V)) \rightarrow \cdots. \end{aligned}$$

Lemma 2.1. *Let M be a manifold and let \mathfrak{h} be a Lie algebra. Suppose that \mathcal{F} is a sheaf of C^∞ -modules and of \mathfrak{h} -modules on M , where the two module structures are compatible,*

i.e. \mathfrak{h} acts compatibly on the sheaf C^∞ by derivations. Assume that U, V are open sets, and that $\phi_U, \phi_V \in C^\infty(M)$ form an \mathfrak{h} -invariant partition of unity for the cover $\{U, V\}$ of $U \cup V$. If the invariant functor $(-)^{\mathfrak{h}}$ is applied to the standard exact sequence

$$0 \rightarrow \mathcal{F}(U \cup V) \rightarrow \mathcal{F}(U) \oplus \mathcal{F}(V) \rightarrow \mathcal{F}(U \cap V) \rightarrow 0$$

the result is an exact sequence.

Proof: Note that the left exactness of the standard sequence is just the sheaf axiom, but the surjectivity of the last map is not true for a general sheaf unless it is a fine sheaf (i.e. has the partition of unity property for sheaves.) Since \mathcal{F} is assumed to be a C^∞ sheaf, the exactness of our standard sequence is guaranteed.

Since the invariant functor $(-)^{\mathfrak{h}}$ is left exact, applying it to the standard sequence yields a left exact sequence. So it remains to show that

$$\mathcal{F}(U)^{\mathfrak{h}} \oplus \mathcal{F}(V)^{\mathfrak{h}} \rightarrow \mathcal{F}(U \cap V)^{\mathfrak{h}} \quad (2.1)$$

is onto. Since the C^∞ and \mathfrak{h} structures on \mathcal{F} are compatible, the map is a C^∞ -module map, i.e. it is compatible with multiplications by functions. Let $a \in \mathcal{F}(U \cap V)^{\mathfrak{h}}$ and let $\{\phi_U, \phi_V\}$ be an \mathfrak{h} -invariant partition of unity subordinate to the cover $\{U, V\}$ of $U \cup V$.

We claim that there is an extension of $(\phi_U|_{U \cap V})a \in \mathcal{F}(U \cap V)^{\mathfrak{h}}$ by zero to all of V (but not to all of U). We have

$$(U \cap V) \cup (V \setminus \text{supp}(\phi_U)) = V.$$

For if $x \in V \setminus (U \cap V)$ then $x \notin \text{supp}(\phi_U) \subset U$, and so x lies on the left side. Now to see that $(\phi_U|_{U \cap V})a \in \mathcal{F}(U \cap V)^{\mathfrak{h}}$ and the zero $0 \in \mathcal{F}(V \setminus \text{supp}(\phi_U))^{\mathfrak{h}}$ glue together to form a section in $\mathcal{F}(V)^{\mathfrak{h}}$, it is enough to check that $(\phi_U|_{U \cap V})a$ restricts to zero on the overlap $W = (U \cap V) \cap (V \setminus \text{supp}(\phi_U))$. This restriction is equal to $(\phi_U|_W)(a|_W)$ since function multiplication commutes with restriction. But $\phi_U|_W = 0$ because $W \cap \text{supp}(\phi_U) = \emptyset$. This proves our claim. Call this extension $a_V \in \mathcal{F}(V)^{\mathfrak{h}}$. Likewise let $a_U \in \mathcal{F}(U)^{\mathfrak{h}}$ be the extension of $(\phi_V|_{U \cap V})a \in \mathcal{F}(U \cap V)^{\mathfrak{h}}$ by zero to all of U .

Now under (2.1) we have

$$(a_V, -a_U) \mapsto (a_V|_{U \cap V}) + (a_U|_{U \cap V}) = (\phi_U|_{U \cap V})a + (\phi_V|_{U \cap V})a = ((\phi_U + \phi_V)|_{U \cap V})a = a$$

since $\phi_U + \phi_V = 1$. This proves that (2.1) is onto. \square

Remark 2.2. *This result holds if we replace “sheaf” with “weak sheaf” since the reconstruction axiom (1.5) holds for finite covers.*

Theorem 2.3. *Let M be a G -manifold and let U, V be G -invariant open sets in M . Then $U \cap V \rightrightarrows U \amalg V \rightarrow U \cup V$ induces a long exact sequence*

$$\cdots \rightarrow \mathbf{H}_G^p(\mathcal{Q}'(U \cup V)) \rightarrow \mathbf{H}_G^p(\mathcal{Q}'(U)) \oplus \mathbf{H}_G^p(\mathcal{Q}'(V)) \rightarrow \mathbf{H}_G^p(\mathcal{Q}'(U \cap V)) \rightarrow \cdots.$$

Proof: Regard $\mathcal{W} = \mathcal{W}(\mathfrak{g})$ as a constant sheaf of vector spaces over M . Then $\mathcal{W} \otimes \mathcal{Q}'$ is a weak sheaf of C^∞ -modules where functions act only on the right factor. It is a weak sheaf of modules over the Lie algebra $\mathfrak{sg}[t]$, as shown in Section 3 of [13]. Choose a partition of unity ϕ_U, ϕ_V of $U \cap V$ as before. Since U, V are G -invariant sets, by averaging over G , we can assume that the two functions are G -invariant. Note that even though $\mathfrak{sg}[t]$ does not act by derivations on a general element of $\mathcal{W} \otimes \mathcal{Q}'$, $\mathfrak{sg}[t]$ does act by derivations on weight zero elements. Moreover, any G -invariant function f , regarded as $1 \otimes f \in \mathcal{W} \otimes \mathcal{Q}'$, is chiral basic (i.e. $\mathfrak{sg}[t]$ -invariant). So ϕ_U, ϕ_V form a $\mathfrak{sg}[t]$ -invariant partition of unity. Hence the preceding lemma can be applied to $\mathcal{F} = \mathcal{W} \otimes \mathcal{Q}'$ and $\mathfrak{h} = \mathfrak{sg}[t]$. The invariant functor applied to the standard sequence for \mathcal{F} yields an exact sequence

$$0 \rightarrow \mathcal{F}(U \cup V)_{bas} \rightarrow \mathcal{F}(U)_{bas} \oplus \mathcal{F}(V)_{bas} \rightarrow \mathcal{F}(U \cap V)_{bas} \rightarrow 0,$$

which induces the corresponding long exact sequence for chiral equivariant cohomology.

\square

The theorem holds if we replace \mathcal{Q}' by \mathcal{Q} . The only tricky part here is that \mathcal{Q} is no longer a C^∞ -module because the Wick product on \mathcal{Q} is not associative. But we still have a \mathbf{C} -bilinear operation $C^\infty \times \mathcal{Q} \rightarrow \mathcal{Q}$, $(f, a) \mapsto fg$: which is a homomorphism of weak sheaves. Moreover, even though \mathcal{Q} is not functorial under general smooth mappings, it is functorial with respect to open inclusions. A partition of unity argument shows that the standard sequence for \mathcal{Q} is still exact. The proof of the preceding lemma then carries over to the case $\mathcal{F} = \mathcal{W} \otimes \mathcal{Q}$.

3. Homotopy Invariance of $\mathbf{H}_G^*(\mathcal{Q}'(M))$

Let M and N be G -manifolds, and let $\phi_0, \phi_1 : M \rightarrow N$ be G -equivariant maps. Let I denote the interval $[0, 1]$, which we regard as a G -manifold equipped with the trivial action.

Definition 3.1. *A G -equivariant homotopy from ϕ_0 to ϕ_1 is a smooth G -equivariant map $\Phi : M \times I \rightarrow N$ such that for all $x \in M$, $\Phi(x, 0) = \phi_0(x)$ and $\Phi(x, 1) = \phi_1(x)$. For each $t \in I$, $\phi_t : M \rightarrow N$ will denote the map $\phi_t(x) = \Phi(x, t)$.*

The main result in this section is

Theorem 3.2. *Let M and N be G -manifolds, and let $\phi_0, \phi_1 : M \rightarrow N$ be G -equivariant maps. If there exists a G -equivariant homotopy Φ from ϕ_0 to ϕ_1 as above, the induced maps $\phi_0^*, \phi_1^* : \mathbf{H}_G^*(\mathcal{Q}'(N)) \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(M))$ are the same.*

We first define an appropriate notion of chiral chain homotopy in the category of $\mathfrak{sg}[t]$ -modules, and show that two morphisms of $\mathfrak{sg}[t]$ -modules which are chiral chain homotopic induce the same map in chiral equivariant cohomology. In the geometric setting, if Φ is a G -equivariant homotopy between $\phi_0, \phi_1 : M \rightarrow N$, we will construct a chiral chain homotopy between the induced maps $\phi_0^*, \phi_1^* : \mathcal{Q}'(N) \rightarrow \mathcal{Q}'(M)$.

An immediate consequence of Theorem 3.2 is that if M is G -equivariantly contractible to a submanifold M' , then $\mathbf{H}_G^*(\mathcal{Q}'(M)) \cong \mathbf{H}_G^*(\mathcal{Q}'(M'))$. By contrast, the functor $\mathbf{H}_G^*(\mathcal{Q}(-))$ does *not* have this property. Let G be simple and let V be a faithful linear representation of G . Then V is G -equivariantly contractible to the origin $o \in V$, but $\mathbf{H}_G^*(\mathcal{Q}(V))_+ = 0$ whereas $\mathbf{H}_G^*(\mathcal{Q}(o))_+ = \mathbf{H}_G^*(\mathbf{C})_+ \neq 0$.

3.1. Chiral chain homotopies

Suppose that \mathcal{A} and \mathcal{B} are $\mathfrak{sg}[t]$ -modules. We define a *chiral chain homotopy* to be a linear map $P : \mathcal{A} \rightarrow \mathcal{B}$, homogeneous of weight 0 and degree -1 , which is G -equivariant and satisfies

$$P\iota_\xi^{\mathcal{A}}(k) + \iota_\xi^{\mathcal{B}}(k)P = 0, \quad PL_\xi^{\mathcal{A}}(k) - L_\xi^{\mathcal{B}}(k)P = 0, \quad (3.1)$$

for all $\xi \in \mathfrak{g}$ and $k \geq 0$.

Lemma 3.3. *If $P : \mathcal{A} \rightarrow \mathcal{B}$ is a chiral chain homotopy, the map $\tau = Pd_{\mathcal{A}} + d_{\mathcal{B}}P$ is a morphism of $\mathfrak{sg}[t]$ -modules.*

Proof: The argument is similar to the proof of Proposition 2.3.1 in [11]. First note that $d_{\mathcal{B}}\tau = d_{\mathcal{B}}Pd_{\mathcal{A}} = \tau d_{\mathcal{A}}$. From the assumption $L_{\xi}^{\mathcal{B}}(k)P = PL_{\xi}^{\mathcal{A}}(k)$, it is immediate that

$$L_{\xi}^{\mathcal{B}}(k)\tau = \tau L_{\xi}^{\mathcal{A}}(k).$$

Finally,

$$\begin{aligned} \iota_{\xi}^{\mathcal{B}}(k)\tau &= \iota_{\xi}^{\mathcal{B}}(k)d_{\mathcal{B}}P + \iota_{\xi}^{\mathcal{B}}(k)Pd_{\mathcal{A}} = \iota_{\xi}^{\mathcal{B}}(k)d_{\mathcal{B}}P - P\iota_{\xi}^{\mathcal{A}}(k)d_{\mathcal{A}} \\ &= -d_{\mathcal{B}}\iota_{\xi}^{\mathcal{B}}(k)P + L_{\xi}^{\mathcal{B}}(k)P + Pd_{\mathcal{A}}\iota_{\xi}^{\mathcal{A}}(k) - PL_{\xi}^{\mathcal{A}}(k) = (d_{\mathcal{B}}P + Pd_{\mathcal{A}})\iota_{\xi}^{\mathcal{A}}(k) = \tau\iota_{\xi}^{\mathcal{A}}(k). \square \end{aligned}$$

We say that two $\mathfrak{sg}[t]$ -module homomorphisms $\phi_0, \phi_1 : \mathcal{A} \rightarrow \mathcal{B}$ are *chiral chain homotopic* if there is a chiral chain homotopy $P : \mathcal{A} \rightarrow \mathcal{B}$ such that $\phi_1 - \phi_0 = \tau$. This clearly implies that ϕ_0, ϕ_1 induce the same map from $\mathbf{H}_{bas}^*(\mathcal{A}) \rightarrow \mathbf{H}_{bas}^*(\mathcal{B})$.

Lemma 3.4. *Chiral chain homotopic maps $\phi_0, \phi_1 : \mathcal{A} \rightarrow \mathcal{B}$ induce the same map from $\mathbf{H}_G^*(\mathcal{A}) \rightarrow \mathbf{H}_G^*(\mathcal{B})$.*

Proof: This is analogous to Proposition 2.4.1 of [11]. The map $id \otimes P : \mathcal{W} \otimes \mathcal{A} \rightarrow \mathcal{W} \otimes \mathcal{B}$ is a chiral chain homotopy between $id \otimes \phi_0, id \otimes \phi_1$. Hence $id \otimes \phi_0, id \otimes \phi_1$ induce the same map from $\mathbf{H}_G^*(\mathcal{A}) = \mathbf{H}_{bas}^*(\mathcal{W} \otimes \mathcal{A}) \rightarrow \mathbf{H}_{bas}^*(\mathcal{W} \otimes \mathcal{B}) = \mathbf{H}_G^*(\mathcal{B})$. \square

Suppose that $\phi_0, \phi_1 : M \rightarrow N$ are G -equivariantly homotopic via $\Phi : M \times I \rightarrow N$. We recall the classical construction of a chain homotopy $P : \Omega^*(N) \rightarrow \Omega^{*-1}(M)$ between $\phi_0^*, \phi_1^* : \Omega^*(N) \rightarrow \Omega^*(M)$, following [11]. For fixed $x \in M$, consider the curve in N given by $s \mapsto \phi_s(x)$, and let $\xi_t : M \rightarrow TN$ be the map which assigns to x the tangent vector to this curve at $s = t$. Consider the map

$$f_t : \Omega^*(N) \rightarrow \Omega^{*-1}(M), \quad \sigma \mapsto \phi_t^*(\iota_{\xi_t}(\sigma)). \quad (3.2)$$

At each $x \in M$, this map is defined by

$$\phi_t^*(\iota_{\xi_t}(\sigma))(\eta_1, \dots, \eta_k) = \sigma(\xi_t(x), d\phi_t(\eta_1), \dots, d\phi_t(\eta_k)), \quad (3.3)$$

given vectors $\eta_1, \dots, \eta_k \in TM_x$.

A well-known formula (see [11]) asserts that

$$\frac{d}{dt}\phi_t^*\sigma = \phi_t^*(\iota_{\xi_t}(d\sigma) + d(\phi_t^*(\iota_{\xi_t}(\sigma))). \quad (3.4)$$

Define $P : \Omega^*(N) \rightarrow \Omega^{*-1}(M)$ by $P\sigma = \int_0^1 \phi_t^*(\iota_{\xi_t}(\sigma)) dt$. Integrating (3.4) over I shows that $Pd + dP = \phi_1^* - \phi_0^*$, so P is a chain homotopy. Since ϕ_0, ϕ_1, Φ are G -equivariant maps, f_t is G -equivariant and satisfies $f_t \iota_{\xi}^N + \iota_{\xi}^M f_t = 0$, for all $t \in I$. It follows that P is also G -equivariant and satisfies $P\iota_{\xi}^N + \iota_{\xi}^M P = 0$. Hence P is a chain homotopy in the sense of [11], and ϕ_0, ϕ_1 induce the same maps in equivariant cohomology.

We need to show that P extends to a map $\mathbf{P} : \mathcal{Q}'(N) \rightarrow \mathcal{Q}'(M)$ which is a chiral chain homotopy between $\phi_0^*, \phi_1^* : \mathcal{Q}'(N) \rightarrow \mathcal{Q}'(M)$. By Lemma 3.2 of [13], for each $m \geq 0$, we may regard $\mathcal{Q}'_M[m]$ as a smooth vector bundle over M of finite rank, which has a local trivialization induced from a collection of charts on M . Given a coordinate open set $U \subset M$ with coordinates $\gamma^1, \dots, \gamma^n$, $U \times V$ is a local trivialization of \mathcal{Q}'_M , where V is the vector space with basis consisting of all nonzero monomials of the form

$$\partial^{r_1} \gamma^{i_1} \dots \partial^{r_k} \gamma^{i_k} \partial^{s_1} c^{j_1} \dots \partial^{s_l} c^{j_l}, \quad (3.5)$$

where $r_1 + \dots + r_k + s_1 + \dots + s_l = m$, and each $r_i > 0$ and $s_i \geq 0$. Here c^i denotes the coordinate one-form $d\gamma^i$.

Let $\pi : M \times I \rightarrow M$ be the projection onto the first factor. We can pull back $\mathcal{Q}'_M[m]$ to a vector bundle $\pi^*(\mathcal{Q}'_M[m]) \rightarrow M \times I$. Let $\Gamma[m] = \Gamma(M \times I, \pi^*(\mathcal{Q}'_M[m]))$ denote the space of smooth sections

$$\sigma : M \times I \rightarrow \mathcal{Q}'_M[m], \quad \sigma(x, t) = (x, v(x, t)),$$

where in local coordinates $v(x, t) = \sum_{j \in J} f_j(\gamma^1, \dots, \gamma^n, t) \mu_j$. Here the set J indexes all monomials of the form (3.5), and each f_j is a smooth function on $U \times I$.

Note that $\Gamma = \bigoplus_{m \geq 0} \Gamma[m]$ is an $\mathfrak{sg}[t]$ -algebra, and that $\frac{d}{dt}$ and ∂ are derivations on Γ . It is clear from the local description of $\sigma(x, t)$ that

$$\frac{d}{dt} \partial(\sigma(x, t)) = \partial\left(\frac{d}{dt} \sigma(x, t)\right).$$

Furthermore, the (fiberwise) integral $\int_0^1 \sigma(x, t) dt$ is a well-defined map from $\Gamma \rightarrow \Gamma(M, \mathcal{Q}'_M)$, and $\int_0^1 \partial \sigma(x, t) dt = \partial \int_0^1 \sigma(x, t) dt$.

3.2. Relative derivations

Let $\phi : A \rightarrow B$ be a map of G^* -algebras. A degree-homogeneous map $f : A \rightarrow B$ is called a *derivation relative to ϕ* , or a ϕ -derivation if

$$f(ab) = f(a)\phi(b) + (-1)^{(\deg f)(\deg a)}\phi(a)f(b), \quad (3.6)$$

for all homogeneous $a, b \in \mathcal{A}$. For example, in the case $A = \Omega(N)$, $B = \Omega(M)$, $\phi = \phi_t^*$, the map f_t given by (3.2) is a ϕ_t^* -derivation.

Similarly, given a morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ of $\mathfrak{sg}[t]$ -algebras, a degree-weight homogeneous linear map $f : \mathcal{A} \rightarrow \mathcal{B}$ will be called a ϕ -derivation if

$$f(a \circ_n b) = f(a) \circ_n \phi(b) + (-1)^{(\deg f)(\deg a)}\phi(a) \circ_n f(b), \quad (3.7)$$

for all homogeneous $a, b \in \mathcal{A}$ and $n \in \mathbf{Z}$. Clearly $f(1) = 0$ and $f(\partial a) = \partial f(a)$ for all $a \in \mathcal{A}$. If \mathcal{A}, \mathcal{B} are abelian vertex algebras, to check that f is a ϕ -derivation, it is enough to show that for all $a, b \in \mathcal{A}$,

$$f(: ab :) = : f(a)\phi(b) : + (-1)^{(\deg f)(\deg a)} : \phi(a)f(b) : , \quad f(\partial a) = \partial f(a). \quad (3.8)$$

Remark 3.5. *A ϕ -derivation f is determined by its values on a set of generators of \mathcal{A} . In the case $\mathcal{A} = \mathcal{Q}'(N)$, $\mathcal{B} = \mathcal{Q}'(M)$, $\phi = \phi_t^*$, since $\mathcal{Q}'(N)$ is generated by $\Omega(N)$, any two ϕ_t^* -derivations which agree on $\Omega(N)$ must agree on all of $\mathcal{Q}'(N)$.*

Remark 3.6. *Suppose that f is a ϕ -derivation and $\delta_{\mathcal{A}}, \delta_{\mathcal{B}}$ are vertex algebra derivations on \mathcal{A}, \mathcal{B} , respectively, which are homogeneous of degree d and satisfy $\phi \circ \delta_{\mathcal{A}} = \delta_{\mathcal{B}} \circ \phi$. Then $f \circ \delta_{\mathcal{A}} - (-1)^{(\deg f)(d)}\delta_{\mathcal{B}} \circ f$ is also a ϕ -derivation.*

For example, for any $\xi \in \mathfrak{g}$, the operators $\iota_{\xi}^{\mathcal{A}}(0), \iota_{\xi}^{\mathcal{B}}(0)$ are vertex algebra derivations of degree -1 , and $L_{\xi}^{\mathcal{A}}(0), L_{\xi}^{\mathcal{B}}(0)$ are vertex algebra derivations of degree 0 . Hence

$$f \circ \iota_{\xi}^{\mathcal{A}}(0) - (-1)^{(\deg f)}\iota_{\xi}^{\mathcal{B}}(0) \circ f, \quad f \circ L_{\xi}^{\mathcal{A}}(0) - L_{\xi}^{\mathcal{B}}(0) \circ f$$

are ϕ_t^* -derivations. If \mathcal{A} and \mathcal{B} are abelian vertex algebras, the operators $\iota_\xi^{\mathcal{A}}(k)$, $\iota_\xi^{\mathcal{B}}(k)$, $L_\xi^{\mathcal{A}}(k)$, $L_\xi^{\mathcal{B}}(k)$ are vertex algebra derivations for all $k \geq 0$, so

$$f \circ \iota_\xi^{\mathcal{A}}(k) - (-1)^{(\deg f)} \iota_\xi^{\mathcal{B}}(k) \circ f, \quad f \circ L_\xi^{\mathcal{A}}(k) - L_\xi^{\mathcal{B}}(k) \circ f$$

are ϕ_t^* -derivations.

Lemma 3.7. *There is a unique extension of the map $f_t : \Omega^*(N) \rightarrow \Omega^{*-1}(M)$ defined by (3.2) to a linear map $F_t : \mathcal{Q}'^*(N) \rightarrow \mathcal{Q}'^{*-1}(M)$, which is a ϕ_t^* -derivation.*

Proof: We first construct F_t locally. On a coordinate open set $U \subset N$ with coordinates $\gamma^1, \dots, \gamma^n$, recall from [13] that $\mathcal{Q}'(U)$ is the abelian vertex algebra with generators $f \in \mathcal{C}^\infty(U)$ and c^i , $i = 1, \dots, n$, subject to the relations:

$$1(z) - 1, \quad (fg)(z) - f(z)g(z), \quad \partial f(z) - \frac{\partial f}{\partial \gamma^i}(z) \partial \gamma^i(z), \quad (3.9)$$

for all $f, g \in \mathcal{C}^\infty(U)$. We define F_t on generators by $F_t(f(\gamma^1, \dots, \gamma^n)) = 0$ and $F_t(c^i) = f_t(c^i)$, and then extend F_t to a linear map on all of $\mathcal{Q}'(U)$ using the ϕ_t^* -derivation property (3.8). Since the relations (3.9) are all homogeneous of degree 0 and F_t lowers degree by one, it is clear that F_t annihilates these relations, and hence is well-defined. Using the fact that $F_t(f(\gamma^1, \dots, \gamma^n)) = 0$ for any $f \in \mathcal{C}^\infty(U)$, it is easy to check that the definition of F_t is coordinate-independent.

Finally, cover N with coordinate open sets $\{U_i\}$, and define $F_t|_{U_i}$ as above. Fix a partition of unity $\{\psi_i\}$ subordinate to this covering. For each $m \geq 0$, we define $F_t : \mathcal{Q}'(N)[m] \rightarrow \mathcal{Q}'(M)[m]$ by $F_t = \sum_i \psi_i F_t|_{U_i}$, which is well-defined since $\mathcal{Q}'(N)[m]$ is a fine sheaf. Moreover, F_t still satisfies (3.8) on each $\mathcal{Q}'(N)[m]$ because $F_t(\psi_i) = 0$. Finally, since $\mathcal{Q}'(N) = \sum_{m \geq 0} \mathcal{Q}'(N)[m]$, this defines F_t on all of $\mathcal{Q}'(N)$. \square

Lemma 3.8. *Equation (3.4) holds for any σ in $\mathcal{Q}'(N)$, not just $\Omega(N)$.*

Proof: By the preceding lemma, both sides of (3.4) are well-defined. It suffices to show that it holds at $t = s$ for each $s \in I$. Let

$$g = \frac{d}{dt} \phi_t^*|_{t=s}(-), \quad h = \phi_s^*(\iota_{\xi_s}(0)d(-)) + d\phi_s^*(\iota_{\xi_s}(0)(-)),$$

which are the maps from $\mathcal{Q}'(N) \rightarrow \mathcal{Q}'(M)$ appearing on the left and right sides of (3.4), evaluated at $t = s$. Clearly g and h are both ϕ_s^* -derivations, and since g and h agree on generators, they agree on all of $\mathcal{Q}'(N)$ by Remark 3.5. \square

We now define $\mathbf{P} : \mathcal{Q}'(N) \rightarrow \mathcal{Q}'(M)$ by $\mathbf{P}(\sigma) = \int_0^1 F_t(\sigma) dt$, which coincides with P at weight zero. Integration of (3.4) over I shows that $d\mathbf{P} + \mathbf{P}d = \phi_1^* - \phi_0^*$. Finally, we need to show that the map \mathbf{P} constructed above is in fact a chiral chain homotopy. Recall that for all $\sigma \in \Omega(N)$, $\xi \in \mathfrak{g}$, and $t \in I$, f_t satisfies

$$f_t L_\xi^N - L_\xi^M f_t = 0, \quad f_t \iota_\xi^N + \iota_\xi^M f_t = 0. \quad (3.10)$$

For $\xi \in \mathfrak{g}$ and $k \geq 0$, consider the maps

$$R_{t,\xi,k} = F_t L_\xi^N(k) - L_\xi^M(k) F_t, \quad S_{t,\xi,k} = F_t \iota_\xi^N(k) + \iota_\xi^M(k) F_t.$$

By Remark 3.6, $R_{t,\xi,k}$ and $S_{t,\xi,k}$ are ϕ_t^* -derivations from $\mathcal{Q}'(N) \rightarrow \mathcal{Q}'(M)$, which are homogeneous of weight $-k$ and degree -1 and -2 , respectively. For $k > 0$, $R_{t,\xi,k}$ and $S_{t,\xi,k}$ both act by zero on $\mathcal{Q}'(N)[0]$ by weight considerations. For $k = 0$, $R_{t,\xi,k}$ and $S_{t,\xi,k}$ act by zero on $\mathcal{Q}'(N)[0]$ by (3.10). Since $R_{t,\xi,k}$ and $S_{t,\xi,k}$ are ϕ_t^* -derivations, it follows from Remark 3.5 that they act by zero on all of $\mathcal{Q}'(N)$.

Finally, since this holds for each $t \in I$, it is immediate that

$$\mathbf{P} L_\xi^N(k)(\sigma) - L_\xi^M(k) \mathbf{P}(\sigma) = \int_0^1 R_{t,\xi,k}(\sigma) dt = 0, \quad \mathbf{P} \iota_\xi^N(k)(\sigma) + \iota_\xi^M(k) \mathbf{P}(\sigma) = \int_0^1 S_{t,\xi,k}(\sigma) dt = 0,$$

for all $\xi \in \mathfrak{g}$, $k \geq 0$ and $\sigma \in \mathcal{Q}'(N)$. Hence \mathbf{P} is a chiral chain homotopy, as desired.

4. A Quasi-conformal Structure on $\mathbf{H}_G^*(\mathcal{Q}'(M))$ and $\mathbf{H}_G^*(\mathcal{Q}(M))$

When G is semisimple, $\mathbf{H}_G^*(\mathbf{C})$ is a conformal vertex algebra with Virasoro element \mathbf{L} of central charge 0. For any M , $\mathbf{H}_G^*(\mathcal{Q}'(M))$ is then a conformal vertex algebra with Virasoro element $\kappa_G(\mathbf{L})$. The vanishing of $\kappa_G(\mathbf{L})$ is a necessary and sufficient condition for the vanishing of $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ since $\kappa_G(\mathbf{L}) \circ_1 \omega = n\omega$ for all $\omega \in \mathbf{H}_G^*(\mathcal{Q}'(M))[n]$. Unfortunately, $\mathbf{H}_G^*(\mathcal{Q}'(M))$ is not a conformal vertex algebra when G has a positive-dimensional center, so a priori we have no such vanishing criterion for general G .

In this section we show that for any G and M , both $\mathbf{H}_G^*(\mathcal{Q}'(M))$ and $\mathbf{H}_G^*(\mathcal{Q}(M))$ have a *quasi-conformal* structure, that is, an action of the subalgebra of the Virasoro algebra generated by $\{L_n \mid n \geq -1\}$, such that $L_{-1} = L_{\circ_0}$ acts by ∂ , and $L_0 = L_{\circ_1}$ acts by $n \cdot id$ on the subspace of weight n . Thus we have a similar vanishing criterion for $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ and $\mathbf{H}_G^*(\mathcal{Q}(M))_+$ for any G ; it suffices to show that L_{\circ_1} acts by zero.

We work in the setting of a general $O(\mathfrak{sg})$ topological vertex algebra (TVA), which we defined in [13]. Recall that an $O(\mathfrak{sg})$ TVA is a degree-weight graded DVA (\mathcal{A}, d) equipped with an $O(\mathfrak{sg})$ -structure $(\xi, \eta) \mapsto L_\xi^{\mathcal{A}} + \iota_\eta^{\mathcal{A}}$, a chiral horizontal element $g^{\mathcal{A}}$, such that $L^{\mathcal{A}} = dg^{\mathcal{A}}$ is a conformal structure, with respect to which $L_\xi^{\mathcal{A}}$ and $\iota_\eta^{\mathcal{A}}$ are primary of weight one. We call $g^{\mathcal{A}}$ a chiral contracting homotopy of \mathcal{A} . Given an $O(\mathfrak{sg})$ TVA (\mathcal{A}, d) , a differential vertex subalgebra \mathcal{B} is called a half $O(\mathfrak{sg})$ TVA if the nonnegative Fourier modes of the vertex operators $\iota_\xi^{\mathcal{A}}$ and $g^{\mathcal{A}}$ preserve \mathcal{B} . Note that the nonnegative Fourier modes of $L_\xi^{\mathcal{A}} = d\iota_\xi^{\mathcal{A}}$ and $L^{\mathcal{A}} = dg^{\mathcal{A}}$ automatically preserve \mathcal{B} as well. In particular, the action of $\{L^{\mathcal{A}} \circ_n \mid n \geq 0\}$ is a quasi-conformal structure on \mathcal{B} . Since $[d, g^{\mathcal{A}} \circ_1] = L^{\mathcal{A}} \circ_1$ and $g^{\mathcal{A}} \circ_1$ acts on \mathcal{B}_{bas} , Theorem 4.8 of [13] shows that $\mathbf{H}_{bas}^*(\mathcal{B})$ vanishes beyond weight zero.

For a G -manifold M , $\mathcal{Q}(M)$ is our main example of an $O(\mathfrak{sg})$ TVA. In local coordinates,

$$g = g^M = b^i \partial \gamma^i, \quad L = L^M = \beta^i \partial \gamma^i - b^i \partial c^i. \quad (4.1)$$

The subalgebra $\mathcal{Q}'(M)$ is then a half $O(\mathfrak{sg})$ TVA as above. Recall that the semi-infinite Weil algebra $\mathcal{W} = \mathcal{W}(\mathfrak{g})$ is *not* an $O(\mathfrak{sg})$ TVA since there is no chiral horizontal element $g^{\mathcal{W}}$ satisfying $dg^{\mathcal{W}} = L^{\mathcal{W}}$.

Let \mathcal{B} be a half $O(\mathfrak{sg})$ TVA inside some $O(\mathfrak{sg})$ TVA \mathcal{A} as above. Then the non-negative Fourier modes of

$$L^{tot} = L^{\mathcal{W}} \otimes 1 + 1 \otimes L^{\mathcal{A}} \in \mathcal{W} \otimes \mathcal{A}$$

act on $\mathcal{W} \otimes \mathcal{B}$, giving $\mathcal{W} \otimes \mathcal{B}$ a quasi-conformal structure. Moreover,

$$L_\xi^{tot} = L_\xi^{\mathcal{W}} \otimes 1 + 1 \otimes L_\xi^{\mathcal{A}}, \quad \iota_\xi^{tot} = \iota_\xi^{\mathcal{W}} \otimes 1 + 1 \otimes \iota_\xi^{\mathcal{A}}$$

are primary of weight one with respect to L^{tot} , and $dL^{tot} = 0$.

Theorem 4.1. *$L^{tot} \circ_n$ operates on $\mathbf{H}_G^*(\mathcal{B})$ for $n \geq 0$, and gives $\mathbf{H}_G^*(\mathcal{B})$ a quasi-conformal structure. If G is semisimple, $\kappa_G(\mathbf{L}) \circ_n = L^{tot} \circ_n$ as operators on $\mathbf{H}_G^*(\mathcal{B})$ for all $n \geq 0$.*

Proof: This is immediate from Theorem 4.8 and Theorem 4.17 of [13]. \square

4.1. A vanishing criterion

For any compact G , and any \mathcal{A} and \mathcal{B} as above, $L^{tot} \circ_1$ acts by $n \cdot id$ on the subspaces $\mathcal{A}[n]$ and $\mathcal{B}[n]$ of weight n . Hence the vanishing of $L^{tot} \circ_1$ on $\mathbf{H}_G^*(\mathcal{B})$ (resp. $\mathbf{H}_G^*(\mathcal{A})$) is equivalent to the vanishing of $\mathbf{H}_G^*(\mathcal{B})_+$ (resp. $\mathbf{H}_G^*(\mathcal{A})_+$). The next lemma gives a useful vanishing criterion for $L^{tot} \circ_1$.

Lemma 4.2. *Suppose that $\alpha \in \mathcal{W} \otimes \mathcal{B}$ is homogeneous of weight 2 and degree -1 , is G -invariant, chiral horizontal, and satisfies*

$$L_\xi^{tot} \circ_1 \alpha = \beta^\xi \otimes 1, \quad (4.2)$$

for all $\xi \in \mathfrak{g}$. Then $L^{tot} \circ_1$ acts by zero on $\mathbf{H}_G^(\mathcal{B})$, and we have $\mathbf{H}_G^*(\mathcal{B})_+ = 0$.*

Proof: Recall from [12] that $d(\beta^{\xi_i} \partial c^{\xi'_i} \otimes 1) = L^\mathcal{W} \otimes 1$, but $\beta^{\xi_i} \partial c^{\xi'_i} \otimes 1$ is not chiral horizontal since $\iota_\xi^\mathcal{W} \circ_n (\beta^{\xi_i} \partial c^{\xi'_i} \otimes 1) = -\delta_{n,1} \beta^\xi \otimes 1$ for $n \geq 0$. Let $\omega_0 = \beta^{\xi_i} \partial c^{\xi'_i} \otimes 1 + d\alpha$. Clearly $d\omega_0 = L^\mathcal{W} \otimes 1$ and

$$\iota_\xi^{tot} \circ_n (d\alpha) = L_\xi^{tot} \circ_n \alpha = \delta_{n,1} \beta^\xi \otimes 1 = -\iota_\xi^{tot} \circ_n (\beta^{\xi_i} \partial c^{\xi'_i} \otimes 1)$$

for $n \geq 0$, since α is chiral horizontal. It follows that $\iota_\xi^{tot} \circ_n \omega_0 = 0$ for all $n \geq 0$, so ω_0 is chiral horizontal. In particular, the non-negative Fourier modes of ω_0 act on $(\mathcal{W} \otimes \mathcal{B})_{bas}$. Finally, let $\omega = \omega_0 + g^\mathcal{A} \in \mathcal{W} \otimes \mathcal{A}$. Since $dg^\mathcal{A} = L^\mathcal{A}$ we have $d\omega = L^{tot}$. The non-negative Fourier modes of ω clearly preserve $(\mathcal{W} \otimes \mathcal{B})_{bas}$ since both ω_0 and $g^\mathcal{A}$ have this property. In particular, $[d, \omega \circ_1] = L^{tot} \circ_1$, so $\omega \circ_1$ is a contracting homotopy for $L^{tot} \circ_1$, as desired. \square

This lemma clearly holds if we replace \mathcal{B} with \mathcal{A} . Note that when G is semisimple, the existence of α as above guarantees that $\kappa_G(\mathbf{L}) = 0$; take $\omega = \beta^{\xi_i} \partial c^{\xi'_i} \otimes 1 + d\alpha + \theta_S^{\xi_i} b^{\xi_i} \otimes 1$. An OPE calculation shows that ω is chiral basic and $d\omega = \kappa_G(\mathbf{L})$.

In [13], we considered two situations where we can construct α as above. First, if G acts locally freely on M , we have a map $\mathfrak{g}^* \rightarrow \Omega^1(M)$ sending $\xi' \rightarrow \theta^{\xi'}$, such that $\iota_\xi \theta^{\eta'} = \langle \eta', \xi \rangle$. The $\theta^{\xi'}$ are known as *connection one-forms*. Choose an orthonormal basis $\{\xi_i\}$ for \mathfrak{g} relative to the Killing form, and let $\Gamma^{\xi'_i} = g \circ_0 \theta^{\xi'_i}$. Then

$$\alpha = \beta^{\xi_i} \otimes \Gamma^{\xi'_i} \in \mathcal{W} \otimes \mathcal{Q}'(M)$$

is G -invariant, chiral horizontal, and satisfies (4.2). This shows that $\mathbf{H}_G^*(\mathcal{Q}'(M))_+ = 0$ and $\mathbf{H}_G^*(\mathcal{Q}(M))_+ = 0$.

Second, let V be a faithful linear representation of G , and let $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ denote the corresponding representation of \mathfrak{g} . The induced bilinear form $\langle \xi, \eta \rangle = \text{Tr}(\rho(\xi)\rho(\eta))$ is nondegenerate, so we may identify \mathfrak{g} with \mathfrak{g}^* via $\langle \cdot, \cdot \rangle$ and fix an orthonormal basis ξ_i of \mathfrak{g} . Let x_k be a basis of V and x'_k the corresponding dual basis for V^* . Define

$$\alpha = \beta^{\xi_i} \otimes \Gamma^{\xi_i} - \beta^{\xi_i} b^{\xi_j} \otimes \iota_{\xi_j} \circ_0 \Gamma^{\xi_i} \in \mathcal{W} \otimes \mathcal{Q}(V), \quad (4.3)$$

where $\Gamma^{\xi_i} = \beta^{\rho(\xi_i)(x_k)} \gamma^{x'_k}$. An OPE calculation shows that α satisfies the conditions of Lemma 4.2, so $\mathbf{H}_G^*(\mathcal{Q}(V))_+ = 0$.

The next lemma shows that locally defined vertex operators α satisfying the conditions of Lemma 4.2 can be glued together.

Lemma 4.3. *Let M be a G -manifold and let $\{U_i \mid i \in I\}$ be a cover of M by G -invariant open sets. Suppose that $\alpha_i \in \mathcal{W} \otimes \mathcal{Q}'(U_i)$ satisfies the conditions of Lemma 4.2. Then $\mathbf{H}_G^*(\mathcal{Q}'(M))_+ = 0$.*

Proof: Let $\{\phi_i \mid i \in I\}$ be a G -invariant partition of unity subordinate to the cover. Let $\alpha = \sum_i \phi_i \alpha_i$, which is a well-defined global section of $\mathcal{W} \otimes \mathcal{Q}'(M)$. Moreover, since ϕ_i is basic, it follows that α remains G -invariant, G -chiral horizontal, and satisfies (4.2). \square

Remark 4.4. *Similarly, if $\alpha_i \in \mathcal{W} \otimes \mathcal{Q}(U_i)$ satisfies the conditions of Lemma 4.2, $\alpha = \sum_i \phi_i \alpha_i \in \mathcal{W} \otimes \mathcal{Q}(M)$ does as well, so that $\mathbf{H}_G^*(\mathcal{Q}(M))_+ = 0$.*

5. $\mathbf{H}_G^*(\mathcal{Q}'(G/H))$ for Homogeneous Spaces G/H

A basic fact about G -manifolds which can be found in [18] is that locally they look like vector bundles over homogeneous spaces G/H .

Theorem 5.1. *Let G be a compact Lie group and let M be a smooth G -manifold. For each point $x \in M$, the isotropy group G_x is a closed subgroup of G and the orbit Gx is G -diffeomorphic to G/G_x . Moreover, Gx has a G -invariant tubular neighborhood U_x which is G -diffeomorphic to the bundle $G \times_{G_x} V$ for some real G_x -representation V .*

By homotopy invariance, $\mathbf{H}_G^*(\mathcal{Q}'(U_x)) \cong \mathbf{H}_G^*(\mathcal{Q}'(G/G_x))$. Thus the problem of computing $\mathbf{H}_G^*(\mathcal{Q}'(G/H))$ for any closed subgroup $H \subset G$ is an important building block in the study of $\mathbf{H}_G^*(\mathcal{Q}'(M))$ for general M .

Suppose first that $K = \text{Ker}(G \rightarrow \text{Diff}(G/H))$ has positive dimension. Clearly $K \subset H$; as in Lemma 1.8, let K_0 denote the identity component of K , and let $G' = G/K_0$ and $H' = H/K_0$. By Lemma 1.8, we have

$$\mathbf{H}_G^*(\mathcal{Q}'(G/H)) = \mathbf{H}_{K_0}^*(\mathbf{C}) \otimes \mathbf{H}_{G'}^*(\mathcal{Q}'(G/H)) = \mathbf{H}_{K_0}^*(\mathbf{C}) \otimes \mathbf{H}_{G'}^*(\mathcal{Q}'(G'/H')).$$

Since G' acts almost effectively on G'/H' , which is a homogeneous space for G' , we may assume without loss of generality that K is finite. In this case, $\mathbf{H}_{K_0}^*(\mathbf{C}) = \mathbf{C}$, so in order to prove Theorem 1.4, it suffices to prove that $\mathbf{H}_G^*(\mathcal{Q}'(G/H))_+ = 0$.

We need a basic property of any simple finite-dimensional complex Lie algebra \mathfrak{g} . Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra of positive codimension. Via the Killing form, we identify \mathfrak{g} with \mathfrak{g}^* , and in particular we identify \mathfrak{h}^\perp with $(\mathfrak{g}/\mathfrak{h})^*$. Note that $(\mathfrak{g}/\mathfrak{h})^*$ is a representation of \mathfrak{h} . Regarding $(\mathfrak{g}/\mathfrak{h})^*$ as a subspace of \mathfrak{g}^* , we may consider the subspace $\text{ad}^*(\mathfrak{g}/\mathfrak{h})^* \subset \mathfrak{g}^*$ under the coadjoint action of \mathfrak{g} on \mathfrak{g}^* .

Lemma 5.2. $\text{ad}^*(\mathfrak{g}/\mathfrak{h})^* = \{\text{ad}_\xi^*(\eta') \mid \xi \in \mathfrak{g}, \eta' \in (\mathfrak{g}/\mathfrak{h})^*\} = \mathfrak{g}^*$.

Proof: If $\mathfrak{h} \subset \mathfrak{h}' \subset \mathfrak{g}$ for some other subalgebra \mathfrak{h}' , we have $(\mathfrak{g}/\mathfrak{h}')^* \subset (\mathfrak{g}/\mathfrak{h})^*$. Then $\text{ad}^*(\mathfrak{g}/\mathfrak{h}')^* \subset \text{ad}^*(\mathfrak{g}/\mathfrak{h})^*$, so we may assume that \mathfrak{h} is a maximal subalgebra of \mathfrak{g} for which $\mathfrak{h} \neq \mathfrak{g}$. If $\text{ad}^*(\mathfrak{g}/\mathfrak{h})^* \neq \mathfrak{g}^*$, there is some nonzero $\xi_0 \in \mathfrak{g}$ such that

$$\langle \xi_0, \text{ad}_{\xi_1}^* \eta' \rangle = 0, \quad \forall \xi_1 \in \mathfrak{g}, \quad \eta' \in (\mathfrak{g}/\mathfrak{h})^* \quad (5.1)$$

Let B denote the set of ξ_0 which satisfy (5.1). Then if $\xi_0 \in B$, for any $\xi_1 \in \mathfrak{g}$ and $\eta' \in (\mathfrak{g}/\mathfrak{h})^*$ we have

$$\langle ad_{\xi_1} \xi_0, \eta' \rangle = 0 \iff ad_{\xi_1} \xi_0 \in \mathfrak{h}. \quad (5.2)$$

Suppose first that $\xi_0 \notin \mathfrak{h}$. Then $\mathfrak{h}' = \mathbf{C}\xi_0 \oplus \mathfrak{h}$ is a Lie subalgebra of \mathfrak{g} . Since \mathfrak{h} is maximal, $\mathfrak{h}' = \mathfrak{g}$. It follows from (5.2) that \mathfrak{h} is a nontrivial ideal of \mathfrak{g} , which contradicts the assumption that \mathfrak{g} is simple. Hence $\xi_0 \in \mathfrak{h}$, so we have $B \subset \mathfrak{h}$.

For any $\xi_1 \in \mathfrak{g}$, we have the decomposition $\xi_1 = \xi_1^{\mathfrak{h}} + \xi_1^{\mathfrak{h}^\perp}$, where $\xi_1^{\mathfrak{h}} \in \mathfrak{h}$ and $\xi_1^{\mathfrak{h}^\perp} \in \mathfrak{h}^\perp$. Note that $ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 = -ad_{\xi_0} \xi_1^{\mathfrak{h}^\perp} \in \mathfrak{h}^\perp$ since $\mathfrak{h}^\perp = (\mathfrak{g}/\mathfrak{h})^*$ is a representation of \mathfrak{h} . But $ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 \in \mathfrak{h}$ by (5.2), so that $ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 \in \mathfrak{h} \cap \mathfrak{h}^\perp$ and we have $ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 = 0$. Hence

$$ad_{\xi_1} \xi_0 = ad_{\xi_1^{\mathfrak{h}}} \xi_0 + ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 = ad_{\xi_1^{\mathfrak{h}}} \xi_0.$$

We claim that for any $\xi_2 \in \mathfrak{g}$ and $\eta' \in (\mathfrak{g}/\mathfrak{h})^*$,

$$\langle ad_{\xi_1} \xi_0, ad_{\xi_2}^* \eta' \rangle = 0,$$

so by definition $ad_{\xi_1} \xi_0 \in B$. Hence B is a nontrivial ideal since $B \neq 0$ and $B \subset \mathfrak{h}$, which is impossible.

We have

$$\langle ad_{\xi_1^{\mathfrak{h}}} \xi_0, ad_{\xi_2}^* \eta' \rangle = -\langle \xi_0, ad_{[\xi_1^{\mathfrak{h}}, \xi_2]}^* \eta' \rangle - \langle \xi_0, ad_{\xi_2}^* ad_{\xi_1^{\mathfrak{h}}}^* \eta' \rangle.$$

The first term above is zero since $\xi_0 \in B$. The second term is also zero since $ad_{\xi_1^{\mathfrak{h}}}^* \eta' \in (\mathfrak{g}/\mathfrak{h})^*$, since $(\mathfrak{g}/\mathfrak{h})^*$ is a representation of \mathfrak{h} . \square

Remark 5.3. *Lemma 5.2 remains true if \mathfrak{g} is semisimple and \mathfrak{h} does not contain any simple component of \mathfrak{g} .*

Proof of Theorem 1.4 for semisimple G : First we assume G is semisimple. Since G acts almost effectively on G/H , \mathfrak{h} does not contain any simple component of \mathfrak{g} , so the conclusion of Lemma 5.2 holds. Fix a basis $\{\xi_1, \dots, \xi_n\}$ of \mathfrak{g} and a corresponding dual basis $\{\xi'_1, \dots, \xi'_n\}$ of \mathfrak{g}^* (relative to the Killing form), such that ξ_1, \dots, ξ_h is a basis

of \mathfrak{h} . By Lemma 4.2, it suffices to construct a G -invariant, G -chiral horizontal element $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)$ satisfying $L_\xi^{tot} \circ_1 \alpha = \beta^\xi \otimes 1$ for all $\xi \in \mathfrak{g}$.

In order to study G/H as a G -space under left multiplication, it is convenient to regard G as a $G \times H$ -space, on which G acts on the left and H acts on the right. The right H -action induces compatible actions of H and $\mathfrak{sh}[t]$ on $\mathcal{Q}'(G)$ which commute with the actions of G and $\mathfrak{sg}[t]$ coming from the left G -action. By Lemma 3.9 of [13], the projection $\pi : G \rightarrow G/H$ induces an isomorphism of vertex algebras

$$\pi^* : \mathcal{Q}'(G/H) \rightarrow \mathcal{Q}'(G)_{H-bas}.$$

Moreover, by declaring that H and $\mathfrak{sh}[t]$ act trivially on $\mathcal{W}(\mathfrak{g})$, we may extend the actions of H and $\mathfrak{sh}[t]$ to $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G)$. We identify the complexes $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)$ and $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G)_{H-bas}$ and regard $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)_{H-bas}$ as a subcomplex of $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G)$. Thus in order to prove Theorem 1.4, it suffices to find a G -invariant, G -chiral horizontal element $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G)_{H-bas}$ satisfying $L_\xi^{tot} \circ_1 \alpha = \beta^\xi \otimes 1$ for all $\xi \in \mathfrak{g}$. In order to deal with all the operators $L_\xi^{tot} \circ_1$ simultaneously, it is convenient to define a new operator

$$\mathcal{L} : \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G) \rightarrow \mathfrak{g} \otimes \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G)$$

sending $\omega \mapsto \xi_k \otimes L_{\xi_k}^{tot} \circ_1 \omega$. Clearly \mathcal{L} is G -equivariant, and the condition $L_\xi^{tot} \circ_1 \alpha = \beta^\xi \otimes 1$ for all $\xi \in \mathfrak{g}$ is equivalent to $\mathcal{L}(\alpha) = \xi_k \otimes \beta^{\xi_k} \otimes 1$.

Let $f \in \mathcal{C}^\infty(G)$ be a smooth function, and fix $\zeta \in \mathfrak{g}$, $\xi \in \mathfrak{g}$, and $\eta' \in \mathfrak{g}^*$. Then for $k = 1, \dots, n$ we have

$$L_{\xi_k}^{tot} \circ_1 (\beta^\zeta b^\xi c^{\eta'} \otimes f) = L_{\xi_k}^{\mathcal{W}} \circ_1 (\beta^\zeta b^\xi c^{\eta'} \otimes f) = \beta^\zeta \otimes \langle [\xi_k, \xi], \eta' \rangle f = \beta^\zeta \otimes \langle \xi_k, ad_\xi^* \eta' \rangle f. \quad (5.3)$$

By Lemma 5.2, there exist elements $\chi_i \in \mathfrak{g}$ and $\eta'_i \in (\mathfrak{g}/\mathfrak{h})^*$ for which $ad_{\chi_i}^* \eta'_i = \xi'_i$, for $i = 1, \dots, n$. Then

$$L_{\xi_k}^{tot} \circ_1 \left(\sum_i \beta^{\xi_i} b^{\chi_i} c^{\eta'_i} \otimes 1 \right) = \beta^{\xi_i} \otimes \langle \xi_k, \xi'_i \rangle = \beta^{\xi_k} \otimes 1,$$

so that

$$\mathcal{L} \left(\sum_i \beta^{\xi_i} b^{\chi_i} c^{\eta'_i} \otimes 1 \right) = \xi_k \otimes \beta^{\xi_k} \otimes 1. \quad (5.4)$$

However, $\sum_i \beta^{\xi_i} b^{\chi_i} c^{\eta'_i} \otimes 1$ is not G -invariant. We seek a G -invariant element

$$\alpha_0 = \sum_{j,k,l} \beta^{\xi_j} b^{\xi_k} c^{\xi'_l} \otimes f_{jkl}$$

which also satisfies (5.4).

We will construct α_0 using the connections one-forms coming from both the left and right actions of G on itself, which we denote by $\theta^{\xi'}$, $\bar{\theta}^{\xi'}$, respectively, for $\xi' \in \mathfrak{g}^*$. We denote the $\mathfrak{sg}[t]$ -algebra structure on $\mathcal{Q}'(G)$ coming from the *right* G -action by $(\xi, \eta) \mapsto \bar{L}_\xi + \bar{l}_\eta$. Evaluating the functions $\iota_\xi \circ_0 \bar{\theta}^{\xi'}$ and $\bar{l}_\xi \circ_0 \theta^{\xi'}$ at the identity $e \in G$, we have

$$\iota_\xi \circ_0 \bar{\theta}^{\xi'}|_e = \langle \xi, \xi' \rangle = \bar{l}_\xi \circ_0 \theta^{\xi'}|_e. \quad (5.5)$$

Define

$$\alpha_0 = \sum_{i,j,k,l} \beta^{\xi_j} b^{\xi_k} c^{\xi'_l} \otimes \bar{l}_{\xi_i}(\theta^{\xi'_j}) \bar{l}_{\chi_i}(\theta^{\xi'_k}) \iota_{\xi_l}(\bar{\theta}^{\eta'_i}).$$

Clearly α_0 is G -invariant, and $\alpha_0|_e = \sum_i \beta^{\xi_i} b^{\chi_i} c^{\eta'_i} \otimes 1$, by (5.5). Acting by \mathcal{L} we see that $(\mathcal{L}(\alpha_0))|_e = \xi_k \otimes \beta^{\xi_k} \otimes 1$. Finally, since α_0 is G -invariant and the operator \mathcal{L} is G -equivariant, it follows that $\mathcal{L}(\alpha_0) = \xi_k \otimes \beta^{\xi_k} \otimes 1$ at every point of G , as desired.

Our next step is to correct α_0 to make it G -chiral horizontal without destroying G -invariance or condition (5.4). Note that for $r \geq 0$,

$$\iota_{\xi_t}^{tot} \circ_r \alpha_0 = b^{\xi_t} \circ_r \alpha_0 = \delta_{r,0} \sum_{i,j,k} \beta^{\xi_j} b^{\xi_k} \otimes \bar{l}_{\xi_i}(\theta^{\xi'_j}) \bar{l}_{\chi_i}(\theta^{\xi'_k}) \iota_{\xi_t}(\bar{\theta}^{\eta'_i}).$$

Let

$$\alpha_1 = - \sum_{i,j,k,l} \beta^{\xi_j} b^{\xi_k} \otimes \bar{l}_{\xi_i}(\theta^{\xi'_j}) \bar{l}_{\chi_i}(\theta^{\xi'_k}) \iota_{\xi_l}(\bar{\theta}^{\eta'_i}) \theta^{\xi'_l}.$$

An OPE calculation shows that for $r \geq 0$

$$L_{\xi_t}^{tot} \circ_r \alpha_1 = 0, \quad \iota_{\xi_t}^{tot} \circ_r \alpha_1 = -\delta_{r,0} \sum_{i,j,k} \beta^{\xi_j} b^{\xi_k} \otimes \bar{l}_{\xi_i}(\theta^{\xi'_j}) \bar{l}_{\chi_i}(\theta^{\xi'_k}) \iota_{\xi_t}(\bar{\theta}^{\eta'_i}). \quad (5.6)$$

Let $\alpha = \alpha_0 + \alpha_1$. It follows from (5.6) that α is G -invariant, G -chiral horizontal, and satisfies $\mathcal{L}(\alpha) = \xi_k \otimes \beta^{\xi_k} \otimes 1$.

We need to correct α so that it lies in $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G)_{H-bas}$, without destroying the above properties. First, we claim that α_0 is already H -chiral horizontal. This is clear since

α is a sum of terms of the form $\beta^{\xi_j} b^{\xi_k} c^{\xi'_l} \otimes f_{jkl}$ where $f_{jkl} \in \mathcal{C}^\infty(G)$, and $\bar{\iota}_\xi \circ_r$ lowers degree and only acts on the second factor of $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G)$ for $\xi \in \mathfrak{h}$.

Second, we claim that α_1 is H -chiral horizontal as well. First note that for $\xi \in \mathfrak{h}$, $\bar{\iota}_\xi \circ_r$ acts by derivations on $\mathcal{Q}'(G)$ for all $r \geq 0$, so it suffices to show that it acts by zero on each term of the form $\bar{\iota}_{\xi_i}(\theta^{\xi'_j})$, $\bar{\iota}_{\chi_i}(\theta^{\xi'_k})$ and $\iota_{\xi_l}(\bar{\theta}^{\eta'_i})\theta^{\xi'_l}$. Clearly $\bar{\iota}_\xi \circ_r$ acts by zero on $\bar{\iota}_{\xi_i}(\theta^{\xi'_j})$ and $\bar{\iota}_{\chi_i}(\theta^{\xi'_k})$ since these terms have degree 0 and $\bar{\iota}_\xi \circ_r$ has degree -1 . Next, note that for each $\eta'_i \in (\mathfrak{g}/\mathfrak{h})^*$ we have

$$\iota_{\xi_l}(\bar{\theta}^{\eta'_i})\theta^{\xi'_l} = \bar{\theta}^{\eta'_i}.$$

which can be checked by applying ι_η , $\eta \in \mathfrak{g}/\mathfrak{h}$ to both sides. Since $\bar{\iota}_\xi \circ_r \bar{\theta}^{\eta'_i} = 0$ for all $\xi \in \mathfrak{h}$ and $\eta'_i \in (\mathfrak{g}/\mathfrak{h})^*$, the claim follows.

Finally, if α is not H -invariant, we can make it H -invariant by averaging it over H , that is, we take $\alpha' = \frac{1}{|H|} \int_H h\alpha \, d\mu$, where $d\mu$ is the Haar measure on H . Since the G and H actions commute, α' is G -invariant and G -chiral horizontal, and $\mathcal{L}(\alpha') = \xi_k \otimes \beta^{\xi_k} \otimes 1$. Moreover, since α is H -chiral horizontal, it follows that $h\alpha$ is still H -chiral horizontal for all $h \in H$, so α' is H -chiral horizontal as well. Since α' is H -invariant and lives in $\mathcal{W}(\mathfrak{g})[2] \otimes \mathcal{Q}'(G)[0]$, α' is in fact H -chiral invariant, so that $\alpha' \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G)_{H\text{-}bas}$, as desired. \square

Proof of Theorem 1.4 for general G : As in Lemma 1.9, let \tilde{G} be a finite cover of G of the form $K \times T$, where K is semisimple and T is a torus. Then

$$\mathbf{H}_G^*(\mathcal{Q}'(G/H)) \cong \mathbf{H}_{\tilde{G}}^*(\mathcal{Q}'(G/H)) \cong \mathbf{H}_{\tilde{G}}^*(\mathcal{Q}'(\tilde{G}/\tilde{H})).$$

Here \tilde{H} is the inverse image of H under the projection $\tilde{G} \rightarrow G$, which is a finite cover of H . Hence without loss of generality we may assume that G is already of the form $K \times T$.

Since G acts almost effectively on G/H , $H \cap T$ is finite, and since T and H commute, H is a normal subgroup of HT . The group HT/H is a torus, which we denote by T' . The natural map $T \rightarrow T'$ is a finite cover, and we have a principal T' -bundle $G/H \rightarrow G/HT$.

Since K is semisimple and K acts almost effectively on

$$G/HT \cong (K \times T)/HT \cong K/(HT \cap K),$$

we can apply Theorem 1.4 in the semisimple case; there exists a K -chiral horizontal, K -invariant element $\alpha \in \mathcal{W}(\mathfrak{k}) \otimes \mathcal{Q}'(G/HT)$ such that $L_\xi^{tot} \circ_1 \alpha = \beta^\xi \otimes 1$ for $\xi \in \mathfrak{k}$.

Since G/H is a T' -bundle over G/HT , the projection $G/H \rightarrow G/HT$ induces an isomorphism $\mathcal{Q}'(G/HT) \rightarrow \mathcal{Q}'(G/H)_{T'-bas}$. Hence we have an injection $\mathcal{W}(\mathfrak{k}) \otimes \mathcal{Q}'(G/HT) \hookrightarrow \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)$. Let α_K denote the image of α under this map. Clearly α_K is invariant under $L_\eta^{G/H} \circ_r$ and $\iota_\eta^{G/H} \circ_r$ for $\eta \in \mathfrak{t}$ and $r \geq 0$ because α_K is a sum of terms of the form $\alpha_i \otimes \omega_i$ with $\omega_i \in \mathcal{Q}'(G/H)_{T'-bas}$. Similarly, $L_\eta^{\mathcal{W}} \circ_r$ and $\iota_\eta^{\mathcal{W}} \circ_r$ act trivially on α_K for $\eta \in \mathfrak{t}$ and $r \geq 0$ since each α_i does not depend on $b^\eta, c^{\eta'}, \beta^\eta, \gamma^{\eta'}$ for $\eta \in \mathfrak{t}', \eta' \in (\mathfrak{t}')^*$. Hence α_K is T' -chiral basic. If α_K is not T -invariant, we can make it T -invariant by averaging it over the finite group $\Gamma = Ker(T \rightarrow T')$.

Clearly α_K is G -chiral horizontal and G -invariant, and satisfies $L_\xi^{tot} \circ_1 \alpha_K = \beta^\xi \otimes 1$ for $\xi \in \mathfrak{k}$ and $L_\eta^{tot} \circ_1 \alpha_K = 0$ for $\eta \in \mathfrak{t}$. We will construct another element $\alpha_T \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)$ which is G -invariant, G -chiral horizontal, and satisfies $L_\xi^{tot} \circ_1 \alpha_T = 0$ for $\xi \in \mathfrak{k}$ and $L_\eta^{tot} \circ_1 \alpha_T = \beta^\eta \otimes 1$ for $\eta \in \mathfrak{t}$. Then $\alpha_K + \alpha_T$ satisfies all the conditions of Lemma 4.2.

Since $H \cap T$ is finite, the action of T on G/H is locally free. Then there are connection forms $\theta^{\eta'} \in \Omega^1(G/H)$ satisfying $\iota_\eta \theta^{\eta'} = \langle \eta', \eta \rangle$. Let $\Gamma^{\eta'} = g \circ_0 \theta^{\eta'}$, which has degree zero and weight one. Let η_i be an orthonormal basis of \mathfrak{t} (relative to the Killing form), and define

$$\alpha = \beta^{\eta_i} \otimes \Gamma^{\eta_i}.$$

Clearly α is T -invariant, T -chiral horizontal, and satisfies $L_\eta^{tot} \circ_1 \alpha = \beta^\eta$ for $\eta \in \mathfrak{t}$. Since K and T commute, we can average α over K without destroying the above properties to make it G -invariant.

We claim that α is in fact G -chiral horizontal. It is enough to check this in local coordinates γ^i on G/H . For $\xi' \in \mathfrak{k}^*$, $\Gamma^{\xi'}$ is of the form $f^k \partial \gamma^k$, where the f^k are smooth functions. Similarly, for each $\xi \in \mathfrak{k}$, the vertex operator $\iota_\xi^{G/H}$ is locally of the form $f^k b^k$. Hence $\Gamma^{\xi'}$ commutes with $\iota_\xi^{G/H}$. Since α does not depend on $c^{\xi'}$ for $\xi' \in \mathfrak{k}^*$, it follows that α commutes with b^ξ for $\xi \in \mathfrak{k}$. Hence $\iota_\xi^{tot} = b^\xi + \iota_\xi^{G/H}$ commutes with α for $\xi \in \mathfrak{k}$, as claimed.

However, α need not satisfy $L_\xi^{tot} \circ_1 \alpha = 0$ for $\xi \in \mathfrak{k}$. The last step is to correct α so that this property holds. Let

$$\alpha_T = \alpha + \theta_S^{\xi_i} L_{\xi_i}^{tot} \circ_1 \alpha,$$

where i runs over a basis of \mathfrak{k} . An OPE calculation using the fact that $L_{\xi_k}^{\mathcal{W}} \circ_1 \theta_S^{\xi_i} = -\delta_{i,k}$ shows that $L_{\xi}^{tot} \circ_1 \alpha_T = 0$ for $\xi \in \mathfrak{k}$. Finally, since $L_{\xi}^{tot} \circ_1$ and $L_{\eta}^{tot} \circ_1$ commute for $\xi \in \mathfrak{k}$ and $\eta \in \mathfrak{t}$, it follows that for all $\eta \in \mathfrak{t}$ we have

$$L_{\eta}^{tot} \circ_1 (\theta_S^{\xi_i} L_{\xi_i}^{tot} \circ_1 \alpha) = \theta_S^{\xi_i} L_{\xi_i}^{tot} \circ_1 (L_{\eta}^{tot} \circ_1 \alpha) = \theta_S^{\xi_i} L_{\xi_i}^{tot} \circ_1 (\beta^{\eta} \otimes 1) = 0.$$

Thus $L_{\eta}^{tot} \circ_1 \alpha_T = L_{\eta}^{tot} \circ_1 \alpha = \beta^{\eta} \otimes 1$, as desired. \square

5.1. Finite-dimensionality of $\mathbf{H}_G^*(\mathcal{Q}'(M))$ for compact M

The subspace $\mathcal{W}(\mathfrak{g})^p[n] \subset \mathcal{W}(\mathfrak{g})$ of degree p and weight n is finite-dimensional, so $\mathbf{H}_G^p(\mathbf{C})[n]$ is finite-dimensional for all $p \in \mathbf{Z}$ and $n \geq 0$. Similarly, since G/H is compact for any closed subgroup $H \subset G$, Theorem 1.4 implies that $\mathbf{H}_G^p(\mathcal{Q}'(G/H))[n]$ is finite-dimensional for all p, n . In this section, we show that if we replace G/H with an arbitrary compact M , $\mathbf{H}_G^p(\mathcal{Q}'(M))[n]$ is finite-dimensional for all p, n as well. This generalizes a well-known classical result in the case $n = 0$. Hence the generating function

$$\chi(G, M) = \sum_{p, n} \dim \mathbf{H}_G^p(\mathcal{Q}'(M))[n] z^p q^n$$

is a well-defined invariant of the G -manifold M .

Lemma 5.4. *If M has a finite cover $\{U_1, \dots, U_m\}$ of G -invariant open sets, such that*

$$\dim \mathbf{H}_G^p(\mathcal{Q}'(U_{i_1} \cap \dots \cap U_{i_k}))[n] < \infty,$$

for each p, n and for fixed indices i_1, \dots, i_k , then $\mathbf{H}^p(\mathcal{Q}'(M))[n]$ is finite-dimensional.

Proof: This is the standard generalized Mayer-Vietoris argument by induction on m . If $m = 1$, there is nothing to prove. For $m = 2$, this is the usual Mayer-Vietoris argument. Put $V = U_1 \cup \dots \cup U_{m-1}$. Then we have

$$\dots \rightarrow \mathbf{H}_G^{p-1}(\mathcal{Q}'(V \cap U_m)) \rightarrow \mathbf{H}_G^p(\mathcal{Q}'(M)) \rightarrow \mathbf{H}_G^p(\mathcal{Q}'(V)) \oplus \mathbf{H}_G^p(\mathcal{Q}'(U_m)) \rightarrow \dots,$$

which we restrict to a given weight n . By inductive hypothesis, $\mathbf{H}_G^p(\mathcal{Q}'(V))[n]$ and $\mathbf{H}_G^p(\mathcal{Q}'(U_m))[n]$ are finite-dimensional, so the third term is finite dimensional. Note that

$V \cap U_m$ is covered by the open sets $U_i \cap U_m$, $i = 1, \dots, m-1$, and their multiple intersections also have finite-dimensional cohomology at fixed p, n . Since there are $m-1$ such open sets, the inductive hypothesis can be applied again. Thus the first term is also finite-dimensional, implying that the second term is finite-dimensional as well. \square

Lemma 5.5. *Suppose the G -manifold M is a fiber bundle whose general fiber is G/H . If M is compact, then $\mathbf{H}_G^p(\mathcal{Q}'(M))[n]$ is finite-dimensional.*

Proof: Choose a local trivializing cover of the bundle, which we can further refine to a good cover on the base M/G , i.e., each multiple intersection of open sets is a ball B . The preimage under $M \rightarrow M/G$ of each multiple intersection of the open sets is equivariantly diffeomorphic to $G/H \times B$. So we can cover M by finitely many open sets whose multiple intersections are equivariantly contractible to G/H . Since $\mathbf{H}_G^p(\mathcal{Q}'(G/H))[n]$ is finite-dimensional for each p, n , the claim follows by the preceding lemma. \square

Given a closed subgroup $H \subset G$, let $M_{(H)}$ denote the subset of M consisting of points with isotropy group conjugate to H . $M_{(H)}$ is a closed submanifold of M , which may be regarded as a G/H -fiber bundle over the manifold $M_{(H)}/G$. By the preceding lemma, $\mathbf{H}_G^p(\mathcal{Q}'(M_{(H)}))[n]$ is finite-dimensional for each p, n .

Theorem 5.6. *Suppose M is compact. Then $\mathbf{H}_G^p(\mathcal{Q}'(M))[n]$ is finite-dimensional.*

Proof: Since M is compact, there are only finitely many conjugacy classes (H) for which $M_{(H)}$ is nonempty. For $\dim H > 0$, each such $M_{(H)}$ has a G -invariant tubular neighborhood $U_{(H)}$ which is equivariantly contractible to $M_{(H)}$. M has a finite cover consisting of the $U_{(H)}$ together with the open set U of points with finite isotropy group. Without loss of generality we can shrink each $U_{(H)}$ so it contains only two orbit types: (H) and (e) .

By homotopy invariance, $\mathbf{H}_G^*(\mathcal{Q}'(U_{(H)})) = \mathbf{H}_G^*(\mathcal{Q}'(M_{(H)}))$, which is finite-dimensional at each p, n . The action of G on U is locally free so $\mathbf{H}_G^*(\mathcal{Q}'(U))_+ = 0$. Since the multiple intersections of the $U_{(H)}$ all lie in U , they also have no higher-weight cohomology. It follows from Lemma 5.4 that for $n > 0$, $\mathbf{H}_G^p(\mathcal{Q}'(M))[n]$ is finite-dimensional for all p . For $n = 0$, the finite-dimensionality of $\mathbf{H}_G^*(\mathcal{Q}'(M))[0] = H_G^p(M)$ is classical. \square

6. The Structure of $\mathbf{H}_G^*(\mathcal{Q}(M))$

For any G -manifold M , Theorem 1.5 gives a complete description of $\mathbf{H}_G^*(\mathcal{Q}(M))$ relative to the family of vertex algebras $\mathbf{H}_K^*(\mathbf{C})$ for connected normal subgroups $K \subset G$. In particular, this result shows that $\mathbf{H}_G^*(\mathcal{Q}(M))_+$ is only sensitive to $\text{Ker}(G \rightarrow \text{Diff}(M))$, and in contrast to $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$, it carries no other geometric information about M .

Proof of Theorem 1.5: We may assume that G acts almost effectively on M . As usual, we need to show that the operator $L^{\text{tot}} \circ_1$ coming from the quasi-conformal structure on $\mathbf{H}_G^*(\mathcal{Q}(M))$ acts by zero. By Lemma 4.2, it is enough to construct a chiral horizontal, G -invariant element $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M)$ for which $L_\xi^{\text{tot}} \circ_1 \alpha = \beta^\xi \otimes 1$ for all $\xi \in \mathfrak{g}$. Since M can be covered by G -invariant open sets of the form $G \times_H V$ for some closed subgroup $H \subset G$ and some H -module V , it is enough to construct $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(G \times_H V)$ for any H and V , by Lemma 4.3 and Remark 4.4. Unlike the functor $\mathbf{H}_G^*(\mathcal{Q}'(-))$, $\mathbf{H}_G^*(\mathcal{Q}(-))$ is not a homotopy invariant, and in general $\mathbf{H}_G^*(\mathcal{Q}(G \times_H V)) \neq \mathbf{H}_G^*(\mathcal{Q}(G/H))$.

Fix $M = G \times_H V$, and assume that G acts almost effectively on M . Note that the action of G on the zero-section $G/H \subset M$ need not be almost effective. Let $K = \text{Ker}(G \rightarrow \text{Diff}(G/H))$, and suppose first that K is finite. Then by Theorem 1.4, there exists $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)$ which we can pull back to $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(M) \subset \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M)$ with the same properties.

So assume that K has positive dimension, and let K_0 denote the identity component. By Lemma 1.9, we may assume that G is of the form $G_1 \times \cdots \times G_k \times T$, where the G_i are simple and T is a torus. Any connected normal subgroup of G splits, so K_0 is a product of a subset of the G_i and a subtorus $T' \subset T$. Hence we may write $G = K_0 \times N$ for some N . Likewise, since K_0 is also a normal subgroup of H , we may write $H = K_0 \times L$. Then

$$M = (K_0 \times N) \times_{K_0 \times L} V = N \times_L V,$$

where N acts on the left and K_0 acts only on the factor V , and the actions of K_0 and L on V commute. Let \mathfrak{n} denote the Lie algebra of N . Clearly $\text{Ker}(N \rightarrow \text{Diff}(N/L))$ is finite, so by the proof of Theorem 1.4 there exists an N -invariant, N -chiral horizontal element $\alpha_N \in \mathcal{W}(\mathfrak{n}) \otimes \mathcal{Q}'(N/L)$ such that

$$L_\xi^{\text{tot}} \circ_1 \alpha_N = \beta^\xi \otimes 1$$

for $\xi \in \mathfrak{n}$. Since K_0 acts trivially on $\mathcal{W}(\mathfrak{n}) \otimes \mathcal{Q}'(N/L)$, we can lift α_N to a G -chiral horizontal G -invariant element, also denoted by α_N , in $\mathcal{W}(\mathfrak{n}) \otimes \mathcal{Q}'(N \times_L V) \subset \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(N \times_L V)$ via the map induced by the projection map $N \times_L V \rightarrow N/L$ such that $L_\xi^{tot} \circ_1 \alpha_N = \beta^\xi \otimes 1$ for $\xi \in \mathfrak{n}$. We can view this element as lying in $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(N \times_L V)$.

On the other hand, the linear action of K_0 on V is faithful because the action of G on M is effective. Thus there exists a K_0 -chiral horizontal, K_0 -invariant element $\alpha_K \in \mathcal{W}(\mathfrak{k}) \otimes \mathcal{Q}(V)$ such that $L_\xi^{tot} \circ_1 \alpha_K = \beta^\xi \otimes 1$ for $\xi \in \mathfrak{k}$. Now L acts only on the $\mathcal{Q}(V)$ factor and it commutes with K_0 action. Thus by averaging over L , we may assume that α_K is L -invariant, and still satisfies the same equation.

Recall from (4.3) that $\alpha_K = \beta^{\xi_i} \otimes \Gamma^{\xi_i} - \beta^{\xi_i} b^{\xi_j} \otimes \iota_{\xi_j} \circ_0 \Gamma^{\xi_i}$, where $\Gamma^{\xi_i} = \beta^{\rho(\xi_i)(x_k)} \gamma^{x'_k}$. Clearly Γ^{ξ_i} is itself L -invariant (because the action of L on x_i and dual action on x'_i amounts to only a change of basis in V .) Now observe that any L -invariant element of $\mathcal{Q}(V)$ can be regarded as a global section in $\mathcal{Q}(N \times_L V)$. (This is clear by covering the bundle by open sets $U \times V$ where $U \subset N/L$, and the observing that the transition functions are elements of L .) The element α_K is now N -chiral horizontal and N -invariant in $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(N \times_L V)$ and satisfies $L_\xi^{tot} \circ_1 \alpha_K = 0$ for $\xi \in \mathfrak{n}$.

Finally, $\alpha = \alpha_N + \alpha_K$ is G -chiral horizontal, G -invariant, and satisfies

$$L_\xi^{tot} \circ_1 \alpha_N = \beta^\xi \otimes 1$$

for all $\xi \in \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$. This completes the proof that $\mathbf{H}_G^*(\mathcal{Q}(M))_+ = 0$. \square

6.1. The ideal property of $\mathbf{H}_G^*(\mathcal{Q}(M))_+$ and $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$

An *ideal* in a vertex algebra \mathcal{A} is a linear subspace which is closed under $\alpha \circ_n$ and $\circ_n \alpha$ for all $n \in \mathbf{Z}$ and $\alpha \in \mathcal{A}$. As usual, if $f : \mathcal{A} \rightarrow \mathcal{B}$ is a vertex algebra homomorphism and $\mathcal{I} \in \mathcal{B}$ is an ideal, then $f^{-1}(\mathcal{I})$ is an ideal of \mathcal{A} .

Lemma 6.1. *For any G , $\mathbf{H}_G^*(\mathbf{C})_+$ is an ideal of $\mathbf{H}_G^*(\mathbf{C})$.*

Let V be a faithful representation of G , and consider the chiral Chern-Weil map $\kappa_G : \mathbf{H}_G^*(\mathbf{C}) \rightarrow \mathbf{H}_G^*(\mathcal{Q}(V))$. At weight zero, this is the identity map $S(\mathfrak{g}^*)^G \rightarrow S(\mathfrak{g}^*)^G$

and it vanishes beyond weight zero because $\mathbf{H}_G^*(\mathcal{Q}(V))_+ = 0$. Hence $\mathbf{H}_G^*(\mathbf{C})_+ = \text{Ker}(\kappa_G)$, which is clearly an ideal. \square

In view of Theorem 1.5 and the preceding lemma, it is immediate that $\mathbf{H}_G^*(\mathcal{Q}(M))_+$ is an ideal of $\mathbf{H}_G^*(\mathcal{Q}(M))$ for any M .

Theorem 6.2. *For any G and M , $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ is an ideal of $\mathbf{H}_G^*(\mathcal{Q}'(M))$.*

Proof: Under the map $\phi : \mathbf{H}_G^*(\mathcal{Q}'(M)) \rightarrow \mathbf{H}_G^*(\mathcal{Q}(M))$ induced by the inclusion $\mathcal{Q}'(M) \hookrightarrow \mathcal{Q}(M)$, we have $\mathbf{H}_G^*(\mathcal{Q}'(M))_+ = \phi^{-1}(\mathbf{H}_G^*(\mathcal{Q}(M))_+)$, which is an ideal since $\mathbf{H}_G^*(\mathcal{Q}(M))_+$ is an ideal of $\mathbf{H}_G^*(\mathcal{Q}(M))$. \square

7. The Structure of $\mathbf{H}_G^*(\mathcal{Q}'(M))$

In contrast to $\mathbf{H}_G^*(\mathcal{Q}(M))$, $\mathbf{H}_G^*(\mathcal{Q}'(M))$ typically contains nontrivial geometric information about M beyond weight zero. Our goal in this section is to give a *relative* description of $\mathbf{H}_G^*(\mathcal{Q}'(M))$ in terms of the vertex algebras $\mathbf{H}_K^*(\mathbf{C})$ for connected normal subgroups $K \subset G$, together with certain geometric data about M . We will focus on three special cases: G simple, $G = G_1 \times G_2$ where G_1, G_2 are simple, and G abelian. We first describe $\mathbf{H}_G^*(\mathcal{Q}'(M))$ as a linear space using Mayer-Vietoris sequences together with Theorem 1.4, and then describe the vertex algebra structure of $\mathbf{H}_G^*(\mathcal{Q}'(M))$.

Let $\mathcal{A} = \sum_{n \geq 0} \mathcal{A}[n]$ and $\mathcal{B} = \sum_{n \geq 0} \mathcal{B}[n]$ be weight graded vertex algebras, and let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a weight-preserving vertex algebra homomorphism. Assume that $\mathcal{A}_+ = \sum_{n > 0} \mathcal{A}[n]$ is an ideal of \mathcal{A} .

Lemma 7.1. *Suppose that the restriction of f to \mathcal{A}_+ is injective. Then the vertex algebra structure of \mathcal{A} is uniquely determined by the ring structure of $(\mathcal{A}[0], \circ_{-1})$, the homomorphism f , and the vertex algebra structure of \mathcal{B} .*

Proof: We need to describe the map

$$\circ_n : \mathcal{A}[i] \otimes \mathcal{A}[j] \rightarrow \mathcal{A}[i + j - n - 1] \quad (7.1)$$

for all $i, j \geq 0$ and $n \leq i + j - 1$ in terms of the above data. First, suppose that $n = i + j - 1$. Since \mathcal{A}_+ is an ideal, it follows that (7.1) is zero unless $i = j = 0$ and $n = -1$, which is known by hypothesis. So we may assume that $n < i + j - 1$.

Let $a \in \mathcal{A}[i]$ and $b \in \mathcal{A}[j]$, and suppose that either $i > 0$ or $j > 0$. Since \mathcal{A}_+ is an ideal, $a \circ_n b \in \mathcal{A}_+$. Since $f : \mathcal{A}_+ \rightarrow \mathcal{B}_+$ is injective, f is an isomorphism of \mathcal{A}_+ onto its image $\mathcal{B}'_+ \subset \mathcal{B}_+$, and the inverse map $f^{-1} : \mathcal{B}'_+ \rightarrow \mathcal{A}_+$ is well-defined. It follows that

$$a \circ_n b = f^{-1}(f(a) \circ_n f(b)),$$

which is determined by f and the vertex algebra structure of \mathcal{B} .

Finally, suppose that $i = j = 0$ and $n \leq -2$. Then $a \circ_n b \in \mathcal{A}_+$, so as above, we have $a \circ_n b = f^{-1}(f(a) \circ_n f(b))$. \square

We will apply Lemma 7.1 in the case where $\mathcal{A} = \mathbf{H}_G^*(\mathcal{Q}'(M))$ and \mathcal{B} is another vertex algebra whose structure is known, to determine the vertex algebra structure on $\mathbf{H}_G^*(\mathcal{Q}'(M))$.

7.1. The case where G is simple

Let G be simple and let M be a G -manifold. The inclusion $i : M^G \rightarrow M$ induces a map

$$\mathbf{i}^* : \mathbf{H}_G^*(\mathcal{Q}'(M)) \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(M^G)) \quad (7.2)$$

on chiral equivariant cohomology, whose restriction to the weight zero subspace $\mathbf{H}_G^*(\mathcal{Q}'(M))[0]$ coincides with the classical map $i^* : H_G^*(M) \rightarrow H_G^*(M^G)$.

Proof of Theorem 1.6: We may assume that M^G is non-empty. Let U_0 be a G -invariant tubular neighborhood of M^G and let $U_1 = M \setminus M^G$. It suffices to show that $\mathbf{H}_G^*(\mathcal{Q}'(U_1))_+ = 0$ and $\mathbf{H}_G^*(\mathcal{Q}'(U_0 \cap U_1))_+ = 0$. In this case, we have $\mathbf{H}_G^*(\mathcal{Q}'(M))_+ \cong \mathbf{H}_G^*(\mathcal{Q}'(U_0 \cup U_1))_+$, and since $\mathbf{H}_G^*(\mathcal{Q}'(U_0))_+ \cong \mathbf{H}_G^*(\mathcal{Q}'(M^G))_+$ by homotopy invariance, the claim follows from a Mayer-Vietoris argument.

For each point $x \in U_1$, the isotropy group G_x has positive codimension in G since G is connected. Let U_x be a G -invariant neighborhood of the orbit Gx , which we may take to be a vector bundle of the form $G \times_{G_x} V$ whose zero-section is Gx .

By Theorem 1.4, there exists a G -invariant, G -chiral horizontal element $\alpha_x \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/G_x)$ satisfying (4.2). Via the projection $U_x \rightarrow Gx$, this pulls back to an element $\alpha_{U_x} \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(U_x)$ satisfying the same conditions. Using a G -invariant partition of unity as in Lemma 4.3, we can glue the α_{U_x} together to obtain $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(U_1)$ satisfying these conditions as well. It follows that $\mathbf{H}_G^*(\mathcal{Q}'(U_1))_+ = 0$. Finally, the same argument shows that $\mathbf{H}_G^*(\mathcal{Q}'(U_0 \cap U_1))_+ = 0$.

Next, let $\mathcal{A} = \mathbf{H}_G^*(\mathcal{Q}'(M))$, $\mathcal{B} = \mathbf{H}_G^*(\mathcal{Q}'(M^G))$, and let $f : \mathcal{A} \rightarrow \mathcal{B}$ be the map i^* given by (7.2). The hypothesis of Lemma 7.1 is satisfied, and the ring structure of $(\mathbf{H}_G^*(\mathcal{Q}'(M))[0], \circ_{-1})$ coincides with the ring structure of $(H_G^*(M), \cup)$, which is classical. As a vertex algebra, $\mathbf{H}_G^*(\mathcal{Q}'(M^G)) \cong \mathbf{H}_G^*(\mathbf{C}) \otimes H^*(M^G)$ where $H^*(M^G)$ is regarded as a vertex algebra in which all products except \circ_{-1} are trivial. By Lemma 7.1, this determines the vertex algebra structure of $\mathbf{H}_G^*(\mathcal{Q}'(M))$ uniquely.

Note that $\mathbf{H}_G^*(\mathcal{Q}'(M))_+ = \mathbf{H}_G^*(\mathbf{C})_+ \otimes H^*(M^G)$ may alternatively be described as

$$\mathbf{H}_G^*(\mathbf{C})_+ \otimes_{S(\mathfrak{g}^*)^G} H_G^*(M^G). \quad (7.3)$$

Given $\alpha \in \mathbf{H}_G^*(\mathbf{C})_+$ and $\omega \in H_G^*(M)$, it follows from Theorem 1.6 that

$$\kappa_G(\alpha) \circ_{-1} \omega = \alpha \otimes i^*(\omega) \in \mathbf{H}_G^*(\mathbf{C})_+ \otimes_{S(\mathfrak{g}^*)^G} H_G^*(M^G).$$

Similarly, given $\omega, \nu \in H^*(M^G)$, $\alpha \otimes \omega$ and $\alpha \otimes \nu$ lie in $\mathbf{H}_G^*(\mathbf{C}) \otimes H^*(M^G)$, and

$$(\alpha \otimes \omega) \circ_{-1} (\alpha \otimes \nu) = (\alpha \circ_{-1} \alpha) \otimes (\omega \cup \nu).$$

Hence both the classical restriction map i^* and the ring structure of $H^*(M^G)$ are encoded in the vertex algebra structure of $\mathbf{H}_G^*(\mathcal{Q}'(M))$. \square

7.2. Chiral equivariant cohomology of spheres

In this section we prove Theorem 1.7. Let G be a compact, connected Lie group. A G -manifold M is said to be *equivariantly formal* if the spectral sequence of the fibration

$$M \hookrightarrow (M \times E)/G \rightarrow E/G$$

collapses. If M is equivariantly formal, the map $i^* : H_G^*(M) \rightarrow H_G^*(M^G)$ is injective, and $H_G^*(M) \cong H^*(M) \otimes H_G^*(pt)$ as a module over $H_G^*(pt)$. Moreover, if M is G -equivariantly

formal, M is also K -equivariantly formal for any closed subgroup $K \subset G$. The following result of [9] gives a useful criterion for equivariant formality.

Theorem 7.2. (*Goresky-Kottwitz-MacPherson*) *If the homology $H_*(M, \mathbf{R})$ can be represented by G -invariant cycles, M is equivariantly formal.*

The next theorem we will need is an immediate consequence of results in [16][17] which describe the fixed-point sets of group actions on disks.

Theorem 7.3. (*Oliver*) *Let F be a finite CW-complex. If G is semisimple, there exists a smooth action of G on a closed disk D with fixed point set D^G having the homotopy type of F . If G is a torus, there exists such an action if and only if F is \mathbf{Z} -acyclic.*

If G acts smoothly on an n -dimensional disk D , we may glue together two copies of D along their boundaries to obtain a smooth action of G on the sphere S^n . It is immediate from Theorem 7.2 that a G -sphere is equivariantly formal if and only if it has a G -fixed point. This allows us to construct compact G -manifolds which have the same classical equivariant cohomology, but distinct chiral equivariant cohomology.

Proof of Theorem 1.7: Let G be simple, and let F be a CW-complex with 3 zero-cells and no higher-dimensional cells. Choose an n -dimensional disk D with a smooth G -action such that D^G has the homotopy type of F . Let S_0 be the copy of S^n obtained by gluing together two copies of D along their boundaries. Note that each connected component \mathcal{C} of D^G gives rise to one component of S_0^G (if $\mathcal{C} \cap \partial D \neq \emptyset$), or two components of S_0^G (if $\mathcal{C} \cap \partial D = \emptyset$). Hence $3 \leq c_0 \leq 6$, where c_0 is the number of components of S_0^G .

Given $x \in S_0^G$, we can find a G -invariant ball $B_0 \subset S_0$ containing x , which intersects exactly one component of S_0^G . By removing B_0 from two copies of S_0 and then gluing them together along their boundaries, we obtain an n -dimensional G -sphere S_1 such that S_1^G has either $2c_0 - 2$ components (if $\partial B \cap S_0^G = \emptyset$) or $2c_0 - 1$ components (if $\partial B \cap S_0^G \neq \emptyset$). We continue this process as follows. Assume that n -dimensional G -spheres S_0, S_1, \dots, S_{i-1} have been defined. Let $B_{i-1} \subset S_{i-1}$ be a G -invariant ball intersecting exactly one component of S_{i-1}^G . Define $D_i = S_{i-1} \setminus B_{i-1}$, and let S_i be the sphere obtained by gluing two copies of D_i along their boundaries. We thus obtain a sequence of

n -dimensional spheres S_0, S_1, S_2, \dots with smooth G -actions, such that the number of fixed-point components c_0, c_1, c_2, \dots are all distinct. Since $c_i = \dim H^0(S_i^G)$, it is immediate from Theorem 1.6 that the vertex algebras $\mathbf{H}_G^*(\mathcal{Q}'(S_i))$ are all distinct.

To complete the proof of Theorem 1.7, it remains to show that the classical equivariant cohomology rings $H_G^*(S_i)$ are all isomorphic to $H_G^*(pt)[\omega]/(\omega^2)$. Let T be a maximal torus of G , and let W be the Weyl group, so that $H_G^*(S_i) = H_T^*(S_i)^W$.

By Theorem 7.3, D_i^T is acyclic, and by a Mayer-Vietoris argument, S_i^T has the homology type of a k -dimensional sphere for some $k < n$. In particular, $\dim H^*(S_i^T) = 2$. Next, we claim that S_i^T is connected, so that $k \geq 1$. To see this, note first that D_i^T must be connected, since D_i^T is acyclic and contains more than two points. It suffices to prove that $D_i^T \cap \partial D_i \neq \emptyset$. Suppose that D_i^T lies in the interior of D_i , and note that $S_i \setminus D_i$ must also contain exactly one T -fixed point component. Given a point $x \in D_i^T$, choose a small G -invariant ball $B \subset D_i$ containing x . Since x is not an isolated T -fixed point, we may assume (by taking B sufficiently small) that $D_i^T \cap \partial B \neq \emptyset$. But $S_i \setminus B$ is then a G -invariant disk with more than one T -fixed point component, which is impossible by Theorem 7.2.

Since S_i is G -equivariantly formal (and hence T -equivariantly formal), the map

$$i^* : H_T^*(S_i) \rightarrow H_T^*(S_i^T) = H^*(S_i^T) \otimes H_T^*(pt) \quad (7.4)$$

is injective, and $H_T^*(S_i) \cong H^*(S_i) \otimes H_T^*(pt)$. As an $H_T^*(pt)$ -module, $H_T^*(S_i)$ has generators 1 and ω of degrees 0 and n , respectively. Since $\dim H^*(S_i^T) = 2$, (7.4) becomes an isomorphism after localization along $H_T^*(pt)$. Letting k be the dimension of S_i^T , since $\dim H^k(S_i^T) = 1$, we must have

$$i^*(\omega) \in H^k(S_i^T) \otimes H_T^*(pt).$$

It follows that $i^*(\omega^2) = 0$, so $\omega^2 = 0$, and we have a direct sum decomposition

$$H_T^*(S_i) = H_T^*(pt) \oplus H_T^*(pt)\omega.$$

Next, the Weyl group W must preserve the subspace $H_T^*(pt)\omega \subset H_T^*(S_i)$, so we have

$$H_G^*(S_i) = H_T^*(S_i)^W = H_T^*(pt)^W \oplus (H_T^*(pt)\omega)^W,$$

which is a free module over $H_T^*(pt)^W = H_G^*(pt)$ having generators of degrees 0 and n . Thus $H_G^*(pt)\omega$ contains an element of degree n , which must be ω , so $H_G^*(S_i) \cong H_G^*(pt)[\omega]/(\omega^2)$ for each $i \geq 1$, as claimed. \square

Next, we give examples of *morphisms* $f : M \rightarrow N$ in the category of compact G -manifolds (that is, smooth G -equivariant maps) for which $f^* : H_G^*(N) \rightarrow H_G^*(M)$ is an isomorphism (over \mathbf{Z}), but $\mathbf{H}_G^*(\mathcal{Q}'(M)) \neq \mathbf{H}_G^*(\mathcal{Q}'(N))$.

Let D_0 be an n -dimensional disk with a smooth G -action for which $\emptyset \neq D^G \neq D$, and let S_0 be the corresponding sphere, as above. Fix a point $p \in S_0^G$, and let U be a G -invariant neighborhood of p equipped with a smooth G -equivariant map $\phi : U \rightarrow \mathbf{R}^n$, where G acts linearly on \mathbf{R}^n , $\phi(p) = 0$, and ϕ maps the closure \bar{U} diffeomorphically onto the disk $D_1 = \{x \in \mathbf{R}^n : |x| \leq 1\}$. We identify $D_1/\partial D_1$ with another copy of S^n , which we denote by S_1 . Note that S_1 is a smooth G -manifold and S_1^G is a sphere S^k for some $0 \leq k < n$. Let $q \in S_1^G$ be the point which corresponds to ∂D_1 under the projection $\pi : D_1 \rightarrow S_1$, and let $g = \pi \circ \phi : U \rightarrow S_1$, which is smooth and G -equivariant. Note that g does *not* extend to a smooth map $S_0 \rightarrow S_1$. However, we can construct a new function f which agrees with g in a neighborhood of $p \in U$, which extends smoothly to all of S_0 .

Lemma 7.4. *There exists a smooth, G -equivariant map $f : S_0 \rightarrow S_1$ such that $f = g$ in a neighborhood of $p \in U$, and $f(S_0 \setminus U) = q$. Moreover, f is smoothly (but not equivariantly) homotopic to the identity map $S^n \rightarrow S^n$, so f induces isomorphisms in both singular and equivariant cohomology with \mathbf{Z} -coefficients.*

Proof: As above, we identify U with the open disk $D_1^\circ = \{(x : |x| < 1)\}$. Let $U' \subset U$ be a G -invariant neighborhood of ∂U of the form $\{x \in U : 1 - \epsilon < |x| \leq 1\}$. Clearly $g(U')$ is a G -invariant neighborhood of q , which we identify with another open disk $D_2^\circ = \{y : |y| < 1\}$ equipped with a linear action of G . Note that the radius $|y|$ is a G -invariant function on D_2 .

Choose a smooth function $h : [0, 1] \rightarrow [0, 1]$ such that $h(t) = 0$ for $0 \leq t < \frac{1}{3}$, and $h(t) = 1$ for $\frac{2}{3} < t \leq 1$. Define $f : U \rightarrow S_1$ by $f(x) = g(x)h(|g(x)|)$ for $x \in U'$, and $f(x) = g(x)$ for $x \in U \setminus U'$. Since $h(|y|)$ is G -invariant, f is G -equivariant and since $f = g$ on a neighborhood of $U \setminus U'$, f is smooth. Moreover, since f maps a neighborhood of ∂U to q (which we have identified with $0 \in D_2$), f extends to a smooth, G -equivariant

map $S_0 \rightarrow S_1$ sending $S_0 \setminus U \rightarrow q$, as desired. Finally, the fact that f is homotopic to $id : S^n \rightarrow S^n$ is clear because $S_0 \setminus U$ is smoothly contractible to a point in S_0 . \square

Theorem 7.5. *Suppose that F is a CW-complex consisting of k zero cells and no higher-dimensional cells. Let D be a disk with a smooth G -action such that D^G has the homotopy type of F , and let S_0 and S_1 be as above. If $k \geq 3$, $\mathbf{H}_G^*(\mathcal{Q}'(S_0)) \neq \mathbf{H}_G^*(\mathcal{Q}'(S_1))$.*

Proof: Since S_1^G is a k -dimensional sphere for $0 \leq k < n$, we have $1 \leq \dim H^0(S_1^G) \leq 2$. Since $k \leq \dim H^0(S_0^G) \leq 2k$, we have $\dim H^0(S_0^G) > \dim H^0(S_1^G)$ for $k \geq 3$. The claim then follows from Theorem 1.6. \square

Even though $f : S_0 \rightarrow S_1$ is homotopic to $id : S^n \rightarrow S^n$, this result does not contradict Theorem 3.2 because there is no *equivariant* homotopy between f and id . Thus unlike the classical equivariant cohomology, the functor $\mathbf{H}_G^*(\mathcal{Q}'(-))$ can distinguish G -manifolds M and N which admit a G -equivariant map which is a homotopy equivalence, as long as M and N are not equivariantly homotopic.

7.3. The case $G = G_1 \times G_2$, where G_1, G_2 are simple

For simple G , Theorem 1.6 describes $\mathbf{H}_G^*(\mathcal{Q}'(M))$ in terms of $\mathbf{H}_G^*(\mathbf{C})$ together with classical geometric data. In this section, we give a similar description of $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ in the case $G = G_1 \times G_2$, where G_1 and G_2 are simple groups. As in the case of simple G , we first describe $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ as a linear space, and then describe the vertex algebra structure.

Theorem 7.6. *Let G_1, G_2 be simple and let $G = G_1 \times G_2$. For any G -manifold M , $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ is linearly isomorphic to*

$$\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1}) \bigoplus \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2}) \bigoplus \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G).$$

Let U_1, U_2 be G -invariant tubular neighborhoods of M^{G_1}, M^{G_2} , respectively. If $x \notin U_1 \cup U_2$, its stabilizer G_x contains neither G_1 nor G_2 , so $\mathbf{H}_G^*(\mathcal{Q}'(G/G_x))_+ = 0$.

A Mayer-Vietoris argument then shows that $\mathbf{H}_G^*(\mathcal{Q}'(M))_+ = \mathbf{H}_G^*(\mathcal{Q}'(U_1 \cup U_2))_+$. Since U_1 , U_2 , and $U_1 \cap U_2$ are equivariantly contractible to M^{G_1} , M^{G_2} , and M^G , respectively, we can replace $\mathbf{H}_G^*(\mathcal{Q}'(U_1))_+$, $\mathbf{H}_G^*(\mathcal{Q}'(U_2))_+$, and $\mathbf{H}_G^*(\mathcal{Q}'(U_1 \cap U_2))_+$ with $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+$, $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+$, and $\mathbf{H}_G^*(\mathcal{Q}'(M^G))_+$ in the Mayer-Vietoris sequence

$$\cdots \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(U_1 \cup U_2))_+ \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(U_1))_+ \oplus \mathbf{H}_G^*(\mathcal{Q}'(U_2))_+ \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(U_1 \cap U_2))_+ \rightarrow \cdots,$$

obtaining

$$\cdots \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(M))_+ \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+ \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(M^G))_+ \rightarrow \cdots. \quad (7.5)$$

Lemma 7.7. *The map $\phi : \mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+ \rightarrow \mathbf{H}_G^*(\mathcal{Q}'(M^G))_+$ appearing in (7.5) is surjective. Hence $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ is canonically isomorphic to $\text{Ker}(\phi)$.*

Proof: Let $i_1 : M^G \rightarrow M^{G_1}$, $i_2 : M^G \rightarrow M^{G_2}$, $j_1 : M^{G_1} \rightarrow M$, $j_2 : M^{G_2} \rightarrow M$ denote the obvious inclusion maps. First we need to describe each of the spaces $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+$, $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+$, and $\mathbf{H}_G^*(\mathcal{Q}'(M^G))_+$. Since G acts trivially on M^G , $\mathbf{H}_G^*(\mathcal{Q}'(M^G))_+ = \mathbf{H}_G^*(\mathbf{C})_+ \otimes H^*(M^G)$, which is isomorphic to

$$\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(pt) \otimes H^*(M^G) \bigoplus H_{G_1}^*(pt) \otimes \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G).$$

Similarly, since G_1 acts trivially on M^{G_1} , $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_1})) = \mathbf{H}_{G_1}^*(\mathbf{C}) \otimes \mathbf{H}_{G_2}^*(\mathcal{Q}'(M^{G_1}))$. Hence $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+$ is isomorphic to

$$\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1}) \bigoplus H_{G_1}^*(pt) \otimes \mathbf{H}_{G_2}^*(\mathcal{Q}'(M^{G_1}))_+ \bigoplus \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_2}^*(\mathcal{Q}'(M^{G_1}))_+.$$

Since G_2 is simple, $\mathbf{H}_{G_2}^*(\mathcal{Q}'(M^{G_1}))_+ = \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G)$ by Theorem 1.6. Hence $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+$ is isomorphic to

$$\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1}) \bigoplus H_{G_1}^*(pt) \otimes \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G).$$

Interchanging the roles of G_1 and G_2 , we see that $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+$ is isomorphic to

$$\mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2}) \bigoplus H_{G_2}^*(pt) \otimes \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H^*(M^G).$$

Next, we need to describe the restriction of ϕ to the various summands of $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+$. Clearly ϕ maps the summand $\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1}) \subset \mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+$ to

$$\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(pt) \otimes H^*(M^G) \subset \mathbf{H}_G^*(\mathcal{Q}'(M^G))_+,$$

acting by $id \otimes i_1^*$. (For this to make sense, we need to identify $\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(pt) \otimes H^*(M^G)$ with $\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(pt) \otimes_{H_{G_2}^*(pt)} H_{G_2}^*(M^G)$, as in (7.3)). Also, ϕ acts by id on the remaining summands of $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+$.

Similarly, ϕ maps the summand $\mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2}) \subset \mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+$ to $\mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(pt) \otimes H^*(M^G)$, acting by $-id \otimes i_2^*$. Finally, ϕ acts by $-id$ the identity on the remaining summands of $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+$. The surjectivity of ϕ is now apparent. \square

Proof of Theorem 7.6 : The following notation will be convenient. Since $\mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+$ decomposes as the direct sum of six subspaces

$$\begin{aligned} & \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1}) \bigoplus H_{G_1}^*(pt) \otimes \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G) \\ & \bigoplus \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2}) \bigoplus H_{G_2}^*(pt) \otimes \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H^*(M^G), \end{aligned}$$

an element $\omega \in \mathbf{H}_G^*(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}_G^*(\mathcal{Q}'(M^{G_2}))_+$ can be written uniquely as a 6-tuple $(\omega_1, \dots, \omega_6)$.

Let $\alpha = \sum_i \alpha_i \otimes \omega_i$ be an arbitrary element of $\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1})$. Via $i_1^* : H_{G_2}^*(M^{G_1}) \rightarrow H_{G_2}^*(M^G)$, $\omega_i \mapsto \sum_j p_{ij} \otimes \nu_{ij} \in H_{G_2}^*(pt) \otimes H^*(M^G) = H_{G_2}^*(M^G)$. Let

$$\tilde{\alpha} = \sum_{i,j} \alpha_i \otimes p_{ij} \otimes \nu_{ij} \in \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(pt) \otimes H^*(M^G),$$

and note that $(\alpha, 0, 0, 0, \tilde{\alpha}, 0)$ lies in $Ker(\phi)$ by construction. Since α was arbitrary, the assignment $\alpha \mapsto (\alpha, 0, 0, 0, \tilde{\alpha}, 0)$ identifies $\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1})$ with a linear subspace of $Ker(\phi)$.

Interchanging the roles of G_1 and G_2 , for any $\alpha \in \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2})$ we can find $\tilde{\alpha} \in$ such that $(0, \tilde{\alpha}, 0, \alpha, 0, 0) \in Ker(\phi)$. Hence the assignment $\alpha \mapsto (0, \tilde{\alpha}, 0, \alpha, 0, 0)$ identifies $\mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2})$ with another subspace of $Ker(\phi)$.

Finally, given $\alpha \in \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H^*(M^G)$, $(0, 0, \alpha, 0, 0, \alpha) \in Ker(\phi)$, so the map $\alpha \mapsto (0, 0, \alpha, 0, 0, \alpha)$ identifies $\mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H^*(M^G)$ with another subspace

of $\text{Ker}(\phi)$. Clearly these three subspaces of $\text{Ker}(\phi)$ intersect trivially, and account for all of $\text{Ker}(\phi)$. \square

Next, we use Lemma 7.1 to describe the vertex algebra structure of $\mathbf{H}_G^*(\mathcal{Q}'(M))$. Taking $\mathcal{A} = \mathbf{H}_G^*(\mathcal{Q}'(M))$ and $\mathcal{B} = \mathbf{H}_{G_1}^*(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}_{G_2}^*(\mathcal{Q}'(M^{G_2}))$, and $f : \mathcal{A} \rightarrow \mathcal{B}$ the map

$$\mathbf{H}_G^*(\mathcal{Q}'(M)) \rightarrow \mathbf{H}_{G_1}^*(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}_{G_2}^*(\mathcal{Q}'(M^{G_2}))$$

appearing in (7.5) it is clear that the hypothesis of Lemma 7.1 holds. The ring structure of $\mathbf{H}_G^*(\mathcal{Q}'(M))[0] = H_G^*(M)$ is classical, and the vertex algebra structure of $\mathbf{H}_{G_1}^*(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}_{G_2}^*(\mathcal{Q}'(M^{G_2}))$ may be described completely in terms of $\mathbf{H}_{G_1}^*(\mathbf{C})$, $\mathbf{H}_{G_2}^*(\mathbf{C})$ and classical data because of the identity

$$\mathbf{H}_G^*(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}_G^*(\mathcal{Q}'(M^{G_2})) = \mathbf{H}_{G_1}^*(\mathbf{C}) \otimes \mathbf{H}_{G_2}^*(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}_{G_2}^*(\mathbf{C}) \otimes \mathbf{H}_{G_1}^*(\mathcal{Q}'(M^{G_2})).$$

Since G_1, G_2 are simple, the vertex algebra structures of both $\mathbf{H}_{G_1}^*(\mathcal{Q}'(M^{G_2}))$ and $\mathbf{H}_{G_2}^*(\mathcal{Q}'(M^{G_1}))$ are given by Theorem 1.6. By Lemma 7.1, this uniquely determines the vertex algebra structure of $\mathbf{H}_G^*(\mathcal{Q}'(M))$.

Finally, via the identification of $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ with

$$\mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1}) \bigoplus \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2}) \bigoplus \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G)$$

given by Theorem 7.6, we can now describe all circle products in $\mathbf{H}_G^*(\mathcal{Q}'(m)) = H_G^*(M) \oplus \mathbf{H}_G^*(\mathcal{Q}'(m))_+$. For example,

- Given $\alpha \otimes \omega \in \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1})$ and $\eta \otimes \nu \in \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2})$,
 $(\alpha \otimes \omega) \circ_{-1} (\eta \otimes \nu) = (\alpha \circ_{-1} \eta) \otimes i_1^*(\omega) \cup i_2^*(\nu) \in \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H^*(M^G).$
- Given $a \in H_G^*(M)$ and $\alpha \otimes \omega \in \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1})$,
 $a \circ_{-1} (\alpha \otimes \omega) = \alpha \otimes j_1^*(a) \cup \omega \in \mathbf{H}_{G_1}^*(\mathbf{C})_+ \otimes H_{G_2}^*(M^{G_1}).$
- Given $a \in H_G^*(M)$ and $\alpha \otimes \omega \in \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2})$,
 $a \circ_{-1} (\alpha \otimes \omega) = \alpha \otimes j_2^*(a) \cup \omega \in \mathbf{H}_{G_2}^*(\mathbf{C})_+ \otimes H_{G_1}^*(M^{G_2}).$

Note that the maps i_1^* , i_2^* , j_1^* and j_2^* , as well as the ring structures of $H_{G_1}^*(M^{G_2})$, $H_{G_2}^*(M^{G_1})$, and $H_G^*(M^G)$ are encoded in the vertex algebra structure of $\mathbf{H}_G^*(\mathcal{Q}'(M))$. For general G , we expect $\mathbf{H}_G^*(\mathcal{Q}'(M))$ to depend on the family of vertex algebras $\mathbf{H}_K^*(\mathbf{C})$ for connected normal subgroups $K \subset G$ for which M^K is nonempty, together with the rings $H_{G/K}^*(M^K)$ and all maps $H_{G/K}^*(M^K) \rightarrow H_{G/K'}^*(M^{K'})$ when $K \subset K'$.

7.4. The case where G is a torus T

In this section, we study $\mathbf{H}_G^*(\mathcal{Q}'(M))$ in the case where G is a torus T . Recall from [13] that $\mathbf{H}_T^*(\mathcal{Q}'(M))$ can be computed using the small chiral Weil complex $\mathcal{C} = \langle \gamma, c \rangle \otimes \mathcal{Q}'(M)$, with differential $K(0) \otimes 1 + 1 \otimes d_{\mathcal{Q}}$. Here $\langle \gamma, c \rangle$ is the subalgebra of $\mathcal{W}(\mathfrak{t})$ generated by the $\gamma^{\xi'_i}, c^{\xi'_i}, \xi' \in \mathfrak{t}^*$, and $K(0)$ is the chiral Koszul differential. $\mathbf{H}_T^*(\mathcal{Q}'(M))$ is an abelian vertex algebra, i.e., a supercommutative algebra equipped with a derivation ∂ of degree 0 and weight 1.

First we consider the case $T = S^1$. As usual, the inclusion $i : M^{S^1} \rightarrow M$ induces a vertex algebra homomorphism $\mathbf{i}^* : \mathbf{H}_{S^1}^*(\mathcal{Q}'(M)) \rightarrow \mathbf{H}_{S^1}^*(\mathcal{Q}'(M^{S^1}))$ whose restriction to weight zero coincides with the classical map $i^* : H_{S^1}^*(M) \rightarrow H_{S^1}^*(M^{S^1})$. The next result is analogous to Theorem 1.6 for simple group actions.

Theorem 7.8. *(Positive-weight localization for circle actions) For any S^1 -manifold M , $\mathbf{i}^* : \mathbf{H}_{S^1}^*(\mathcal{Q}'(M))_+ \rightarrow \mathbf{H}_{S^1}^*(\mathcal{Q}'(M^{S^1}))_+$ is an isomorphism of vertex algebra ideals. Hence*

$$\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))_+ \cong \mathbf{H}_{S^1}^*(\mathbf{C})_+ \otimes H^*(M^{S^1}). \quad (7.6)$$

Moreover, both the ring structure of $H^*(M^{S^1})$ and the map $i^* : H_{S^1}^*(M) \rightarrow H_{S^1}^*(M^{S^1})$ are encoded in the vertex algebra structure of $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$.

Proof: Every point in M is either an S^1 -fixed point or has a finite isotropy group. If M^{S^1} is nonempty, fix an S^1 -invariant tubular neighborhood U of M^{S^1} , and let $V = M \setminus M^{S^1}$. Since S^1 acts locally freely on V , $\mathbf{H}_{S^1}^*(\mathcal{Q}'(V))_+ = 0 = \mathbf{H}_{S^1}^*(\mathcal{Q}'(U \cap V))_+$. By a Mayer-Vietoris argument, $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))_+ \cong \mathbf{H}_{S^1}^*(\mathcal{Q}'(U))_+$, and by homotopy invariance $\mathbf{H}_{S^1}^*(\mathcal{Q}'(U))_+ \cong \mathbf{H}_{S^1}^*(\mathcal{Q}'(M^{S^1}))_+$. Finally, since S^1 acts trivially on M^{S^1} , we have $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M^{S^1}))_+ \cong \mathbf{H}_{S^1}^*(\mathbf{C})_+ \otimes H^*(M^{S^1})$.

As for the vertex algebra structure, taking $\mathcal{A} = \mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$, $\mathcal{B} = \mathbf{H}_{S^1}^*(\mathcal{Q}'(M^{S^1}))$ and $f = \mathbf{i}^*$, the hypothesis of Lemma 7.1 is clearly satisfied. The ring structure of $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))[0] = H_{S^1}^*(M)$ is classical, and $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M^{S^1})) = \mathbf{H}_{S^1}^*(\mathbf{C}) \otimes H^*(M^{S^1})$ as a vertex algebra. This determines the vertex algebra structure of $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$. As in the case where G is simple, both the classical restriction map i^* and the ring structure of $H^*(M^{S^1})$ are encoded in the vertex algebra structure of $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$. \square

Recall from Theorem 6.1 of [12] that $\mathbf{H}_{S^1}^*(\mathbf{C})$ is just the polynomial algebra $\mathbf{C}[\gamma, \partial\gamma, \partial^2\gamma, \dots]$. Hence $\mathbf{H}_{S^1}^*(\mathbf{C})_+$ is the ideal $\langle \partial\gamma, \partial^2\gamma, \dots \rangle \subset \mathbf{C}[\gamma, \partial\gamma, \partial^2\gamma, \dots]$. Equivalently, $\mathbf{H}_{S^1}^*(\mathbf{C})_+$ may be described as the vertex algebra ideal generated by $\partial\gamma$. Thus Theorem 7.8 gives a complete description of $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$ in terms of classical data.

Example 7.9. $M = \mathbf{CP}^1$, where S^1 has isotropy weights $-1, 1$ at the fixed points p_0, p_1 .

Since $M^{S^1} = \{p_0, p_1\}$, Theorem 7.8 shows that

$$\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))_+ = \mathbf{H}_{S^1}^*(\mathbf{C})_+ \otimes H^*(\{p_0, p_1\}) = \mathbf{H}_{S^1}^*(\mathbf{C})_+ \oplus \mathbf{H}_{S^1}^*(\mathbf{C})_+. \quad (7.7)$$

It follows that $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$ is the free abelian vertex algebra generated by $H_{S^1}^*(M)$.

Example 7.10. $M = \mathbf{CP}^2$, where S^1 acts with isotropy weights $-1, 0, 1$ at the fixed points.

We claim that $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$ is *not* the vertex algebra $\langle H_{S^1}^*(M) \rangle$ generated by the weight zero component. Classically, $H_{S^1}^*(M) \cong \mathbf{C}[t, \omega] / \langle \omega(\omega - t)(\omega + t) \rangle$, where ω is the equivariant symplectic form and t is the image of the generator of $H_{S^1}^*(pt)$ under the Chern-Weil map. From this description, it is clear that $\dim H_{S^1}^2(M) = 2$. Hence in the vertex subalgebra $\langle H_{S^1}^*(M) \rangle \subset \mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$, the subspace of degree 2 and weight 1 can have dimension at most 2. On the other hand, M^{S^1} consists of three isolated fixed points, so $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))_+ = \mathbf{H}_{S^1}^*(\mathbf{C})_+ \oplus \mathbf{H}_{S^1}^*(\mathbf{C})_+ \oplus \mathbf{H}_{S^1}^*(\mathbf{C})_+$ by Theorem 7.8. In particular, $\dim \mathbf{H}_{S^1}^2(\mathcal{Q}'(M))[1] = 3$, so it must contain elements that do not lie in $\langle H_{S^1}^*(M) \rangle$.

Theorem 7.11. For any S^1 -manifold M , $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$ is generated as a vertex algebra by $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))[0] \oplus \mathbf{H}_{S^1}^*(\mathcal{Q}'(M))[1]$.

Proof: Fix a basis $\{\alpha_i \mid i \in I\}$ of $H^*(M^{S^1})$. Consider the collection

$$\mathcal{C} = \{\partial\gamma \otimes \alpha_i \mid i \in I\} \subset \mathbf{H}_{S^1}^*(\mathbf{C})[1] \otimes H^*(M^{S^1}),$$

and let $\langle \mathcal{C} \rangle$ denote the vertex subalgebra of $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$ generated by \mathcal{C} . Since

$$\mathbf{i}^* : \mathbf{H}_{S^1}^*(\mathcal{Q}'(M)) \rightarrow \mathbf{H}_{S^1}^*(\mathcal{Q}'(M^{S^1})) \cong \mathbf{H}_{S^1}^*(\mathbf{C}) \otimes H^*(M^{S^1})$$

preserves ∂ , it follows that $(\partial^{k+1}\gamma) \otimes \alpha_i = \partial^k(\partial\gamma \otimes \alpha_i)$ lies in $\langle \mathcal{C} \rangle$, for all $k \geq 0$. We claim that \mathcal{C} together with $H_{S^1}^*(M)$ generates $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$ as a vertex algebra. At weight zero the claim is obvious, so let $\omega \in \mathbf{H}_{S^1}^*(\mathcal{Q}'(M))_+$ be a nonzero element. In particular, this implies that M^{S^1} is nonempty.

By Theorem 7.8, we can write $\omega = \sum_{i \in I} p_i \otimes \alpha_i$, where $p_i \in \mathbf{H}_{S^1}^*(\mathbf{C})_+$. Since p_i has positive weight, it is divisible by $\partial^{k_i}\gamma$ for some $k_i > 0$, and we may write $p_i = q_i \partial^{k_i}\gamma$ where $q_i \in \mathbf{H}_{S^1}^*(\mathbf{C})$. Since M^{S^1} is nonempty, the chiral Chern-Weil map is injective, and since $\mathbf{H}_{S^1}^*(\mathbf{C})$ is generated by $H_{S^1}^*(pt)$ as a vertex algebra, $q_i \otimes 1$ lies in $\langle H_{S^1}^*(M) \rangle$. Since $p_i \otimes \alpha_i = : (q_i \otimes 1)(\partial^{k_i}\gamma \otimes \alpha_i) :$ for each $i \in I$, the claim follows. \square

Corollary 7.12. *If M is a compact S^1 -manifold, $\mathbf{H}_{S^1}^*(\mathcal{Q}'(M))$ is finitely generated as a vertex algebra.*

7.5. The case $T = S^1 \times S^1$ and $M = \mathbf{CP}^2$

Next, we consider the case where $T = S^1 \times S^1$. This is analogous to the case $G = G_1 \times G_2$ with G_1, G_2 simple, but can be more subtle because many different copies of S^1 inside T can arise as stabilizers of points in M . As an example, we compute $\mathbf{H}_T^*(\mathcal{Q}'(M))$ in the case $M = \mathbf{CP}^2$ with the usual linear action of T . Note that T contains three copies of S^1 which arise as stabilizer subgroups, which we denote by T_i , $i = 1, 2, 3$. Each M^{T_i} is a copy of \mathbf{CP}^1 which we denote by M_i , and $M_i \cap M_j$ consists of a single point p_{ij} . Let U_i be a T -invariant tubular neighborhood of M_i . Clearly $\mathbf{H}_T^*(\mathcal{Q}'(U_i)) = \mathbf{H}_T^*(\mathcal{Q}'(M_i)) = \mathbf{H}_{T_i}^*(\mathbf{C}) \otimes \mathbf{H}_{T/T_i}^*(\mathcal{Q}'(M_i))$, and the action of T/T_i on M_i is the standard action of S^1 on \mathbf{CP}^1 .

Lemma 7.13. $\mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+$ is linearly isomorphic to

$$\begin{aligned} & \mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes H_{T/T_1}^*(M^{T_1}) \bigoplus \mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes H_{T/T_2}^*(M^{T_2}) \\ & \bigoplus \mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathcal{Q}'(p_{13}))_+ \bigoplus \mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_2}^*(\mathcal{Q}'(p_{23}))_+ \bigoplus \mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathcal{Q}'(p_{12}))_+. \end{aligned} \quad (7.8)$$

Proof: Consider the Mayer-Vietoris sequence

$$\cdots \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+ \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(U_1))_+ \oplus \mathbf{H}_T^*(\mathcal{Q}'(U_2))_+ \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cap U_2))_+ \rightarrow \cdots,$$

which we can replace with

$$\cdots \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+ \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(M_1))_+ \oplus \mathbf{H}_T^*(\mathcal{Q}'(M_2))_+ \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(p_{12}))_+ \rightarrow \cdots. \quad (7.9)$$

By (7.7), $\mathbf{H}_T^*(\mathcal{Q}'(M_1))_+$ is linearly isomorphic to

$$\begin{aligned} & \mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes H_{T/T_1}^*(M_1) \bigoplus H_{T_1}^*(pt) \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{13}\}) \\ & \bigoplus \mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{13}\}). \end{aligned}$$

Likewise, $\mathbf{H}_T^*(\mathcal{Q}'(M_2))_+$ is linearly isomorphic to

$$\begin{aligned} & \mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes H_{T/T_2}^*(M_2) \bigoplus H_{T_2}^*(pt) \otimes \mathbf{H}_{T/T_2}^*(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{23}\}) \\ & \bigoplus \mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_2}^*(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{23}\}). \end{aligned}$$

Thus the middle term $\mathbf{H}_T^*(\mathcal{Q}'(M_1))_+ \oplus \mathbf{H}_T^*(\mathcal{Q}'(M_2))_+$ in (7.9) can be identified with the direct sum of the above six subspaces, so we may write $\omega \in \mathbf{H}_T^*(\mathcal{Q}'(M_1))_+ \oplus \mathbf{H}_T^*(\mathcal{Q}'(M_2))_+$ as a 6-tuple $(\omega_1, \dots, \omega_6)$ as in the proof of Theorem 7.6. The map

$$\mathbf{H}_T^*(\mathcal{Q}'(M_1))_+ \oplus \mathbf{H}_T^*(\mathcal{Q}'(M_2))_+ \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(p_{12}))_+$$

in (7.9), which we denote by ϕ , is surjective, so we may identify $\mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+$ with $\text{Ker}(\phi)$.

Given $\alpha \in \mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes H_{T/T_1}^*(M^{T_1})$, we can find

$$\tilde{\alpha} \in H_{T_2}^*(pt) \otimes \mathbf{H}_{T/T_2}^*(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{23}\})$$

such that $(\alpha, 0, 0, 0, \tilde{\alpha}, 0) \in \text{Ker}(\phi)$. The assignment $\alpha \mapsto (\alpha, 0, 0, 0, \tilde{\alpha}, 0)$ identifies $\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes H_{T/T_1}^*(M^{T_1})$ with a linear subspace of $\text{Ker}(\phi)$. Likewise, given $\alpha \in \mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes H_{T/T_2}^*(M_2)$, there exists

$$\tilde{\alpha} \in H_{T_1}^*(pt) \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{13}\})$$

such that $(0, \tilde{\alpha}, 0, \alpha, 0, 0) \in \text{Ker}(\phi)$, so $\mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes H_{T/T_2}^*(M_2)$ may be identified with another subspace of $\text{Ker}(\phi)$.

Next, note that $H^*({p_{12}, p_{13}}) = H^*(p_{12}) \oplus H^*(p_{13})$, so that $\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*(p_{13})$ may be regarded as a subspace of $\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*({p_{12}, p_{13}})$. Clearly $\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*(p_{13})$ lies in $\text{Ker}(\phi)$ since $p_{13} \notin U_1 \cap U_2$. Similarly, $\mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_2}^*(\mathbf{C})_+ \otimes H^*(p_{23})$ is a subspace of $\mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_2}^*(\mathbf{C})_+ \otimes H^*({p_{12}, p_{23}})$ which lies in the $\text{Ker}(\phi)$. Finally, note that

$$\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ = \mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_2}^*(\mathbf{C})_+.$$

It follows that $\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*({p_{12}, p_{13}})$ and $\mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_2}^*(\mathbf{C})_+ \otimes H^*({p_{12}, p_{23}})$ each contain a copy of $\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*(p_{12})$. Thus given $\alpha \in \mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*(p_{12})$, $(0, 0, \alpha, 0, 0, \alpha)$ will lie in $\text{Ker}(\phi)$. This identifies $\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*(p_{12})$ with another subspace of $\text{Ker}(\phi)$. Finally, it is straightforward to check that these five subspaces of $\text{Ker}(\phi)$ intersect pairwise trivially and account for all of $\text{Ker}(\phi)$. This completes the proof of Lemma 7.13. \square

Next, since T acts locally freely on the complement of $U_1 \cup U_2 \cup U_3$,

$$\mathbf{H}_T^*(\mathcal{Q}'(M))_+ = \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2 \cup U_3))_+,$$

so we have a Mayer-Vietoris sequence

$$\cdots \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(M))_+ \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+ \oplus \mathbf{H}_T^*(\mathcal{Q}'(U_3))_+ \rightarrow \mathbf{H}_T^*(\mathcal{Q}'((U_1 \cup U_2) \cap U_3))_+. \quad (7.10)$$

Note that $(U_1 \cup U_2) \cap U_3 = \{p_{13}, p_{23}\}$, so that

$$\mathbf{H}_T^*(\mathcal{Q}'((U_1 \cup U_2) \cap U_3))_+ = \mathbf{H}_T^*(\mathcal{Q}'(p_{13}))_+ \oplus \mathbf{H}_T^*(\mathcal{Q}'(p_{23}))_+ = \mathbf{H}_T^*(\mathbf{C})_+ \oplus \mathbf{H}_T^*(\mathbf{C})_+.$$

As for the other terms in (7.10), $\mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+$ is given by (7.8), and $\mathbf{H}_T^*(\mathcal{Q}'(U_3))_+$ is isomorphic to

$$\mathbf{H}_{T_3}^*(\mathbf{C})_+ \otimes H_{T/T_3}^*(M_3) \bigoplus H_{T_3}^*(pt) \otimes \mathbf{H}_{T/T_3}^*(\mathcal{Q}'(M_3))_+ \bigoplus \mathbf{H}_{T_3}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_3}^*(\mathcal{Q}'(M_3))_+,$$

where

$$\mathbf{H}_{T/T_3}^*(\mathcal{Q}'(M_3))_+ = \mathbf{H}_{T/T_3}^*(\mathcal{Q}'(p_{13}))_+ \oplus \mathbf{H}_{T/T_3}^*(\mathcal{Q}'(p_{23}))_+ = \mathbf{H}_{T/T_3}^*(\mathbf{C})_+ \oplus \mathbf{H}_{T/T_3}^*(\mathbf{C})_+,$$

by (7.7). It is easy to check that the map

$$\mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+ \oplus \mathbf{H}_T^*(\mathcal{Q}'(U_3))_+ \rightarrow \mathbf{H}_T^*(\mathcal{Q}'((U_1 \cup U_2) \cap U_3))_+$$

in (7.10), which we denote by ψ , is surjective, so that $\mathbf{H}_T^*(\mathcal{Q}'(M))_+ = \text{Ker}(\psi)$.

Suppose that $\alpha \in \mathbf{H}_{T_3}^*(\mathbf{C})_+ \otimes H_{T/T_3}^*(M_3) \subset \mathbf{H}_T^*(\mathcal{Q}'(M_3))_+$. Then there are unique elements

$$\alpha_1 \in \mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes H_{T/T_1}^*(M_1) \subset \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+,$$

$$\alpha_2 \in \mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes H_{T/T_2}^*(M_2) \subset \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+,$$

such that $(\alpha_1 + \alpha_2, \alpha) \in \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+ \oplus \mathbf{H}_T^*(\mathcal{Q}'(M_3))_+$ lies in $\text{Ker}(\psi)$. The assignment $\alpha \mapsto (\alpha_1 + \alpha_2, \alpha)$ identifies $\mathbf{H}_{T_3}^*(\mathbf{C})_+ \otimes H_{T/T_3}^*(M_3)$ with a linear subspace of $\text{Ker}(\psi)$. Similarly, we may identify each of the spaces

$$\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes H_{T/T_1}^*(M_1) \subset \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+,$$

$$\mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes H_{T/T_2}^*(M_2) \subset \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+,$$

$$\mathbf{H}_{T_1}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_1}^*(\mathbf{C})_+ \otimes H^*(p_{12}) \subset \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+,$$

$$\mathbf{H}_{T_2}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_2}^*(\mathbf{C})_+ \otimes H^*(p_{23}) \subset \mathbf{H}_T^*(\mathcal{Q}'(U_1 \cup U_2))_+,$$

$$\mathbf{H}_{T_3}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_3}^*(\mathbf{C})_+ \otimes H^*(p_{13}) \in \mathbf{H}_T^*(\mathcal{Q}'(M_3))_+,$$

with a subspace of $\text{Ker}(\psi)$. It is easy to check that these subspaces intersect pairwise trivially and account for all of $\text{Ker}(\psi)$. To summarize, we have proved

Theorem 7.14. *For $M = \mathbf{CP}^2$ and $T = S^1 \times S^1$ as above, $\mathbf{H}_T^*(\mathcal{Q}'(M))_+$ is linearly isomorphic to*

$$\left(\bigoplus_{i=1}^3 \mathbf{H}_{T_i}^*(\mathbf{C})_+ \otimes H_{T/T_i}^*(M_i) \right) \bigoplus \left(\bigoplus_{i=1}^3 \mathbf{H}_{T_i}^*(\mathbf{C})_+ \otimes \mathbf{H}_{T/T_i}^*(\mathbf{C})_+ \right). \quad (7.11)$$

As in the case $G = G_1 \times G_2$ for G_1, G_2 simple, the vertex algebra structure of $\mathbf{H}_T^*(\mathcal{Q}'(M))$ for $M = \mathbf{CP}^2$ and $T = S^1 \times S^1$ may be deduced from Theorem 7.14. Let $\mathcal{A} = \mathbf{H}_T^*(\mathcal{Q}'(M))$, and $\mathcal{B} = \mathbf{H}_T^*(\mathcal{Q}'(U_1 \oplus U_2)) \oplus \mathbf{H}_T^*(\mathcal{Q}'(U_3))$, and let $f : \mathcal{A} \rightarrow \mathcal{B}$ be the map

$$\mathbf{H}_T^*(\mathcal{Q}'(M)) \rightarrow \mathbf{H}_T^*(\mathcal{Q}'(U_1 \oplus U_2)) \oplus \mathbf{H}_T^*(\mathcal{Q}'(U_3))$$

appearing in (7.10). Clearly the conditions of Lemma 7.1 are satisfied. As usual, the product \circ_{-1} on $\mathbf{H}_T^*(\mathcal{Q}'(M))[0] = H_T^*(M)$ is classical, and the vertex algebra structure of

\mathcal{B} is determined by Lemma 7.13 and the structure of $\mathbf{H}_{S^1}^*(\mathcal{Q}'(\mathbf{CP}^1))$, which is given by (7.7). By Lemma 7.1, the vertex algebra structure of $\mathbf{H}_T^*(\mathcal{Q}'(M))$ is uniquely determined by this data.

Corollary 7.15. *For $M = \mathbf{CP}^2$ and $T = S^1 \times S^1$, $\mathbf{H}_T^*(\mathcal{Q}'(M))$ is generated as a vertex algebra by $\bigoplus_{n=0}^2 \mathbf{H}_T^*(\mathcal{Q}'(M))[n]$.*

Proof: This is immediate from the vertex algebra structure of $\mathbf{H}_T^*(\mathcal{Q}'(M))$ and the structure of $\mathbf{H}_T^*(\mathbf{C})$. \square

For a general torus T , if M is a T -manifold of finite orbit type (i.e., only a finite number of subtori $T' \subset T$ can occur as isotropy groups for points in M), we expect that $\mathbf{H}_T^*(\mathcal{Q}'(M))$ will be generated as a vertex algebra by $\bigoplus_{n=0}^N \mathbf{H}_T^*(\mathcal{Q}'(M))[n]$ for some N . Note that a similar statement for $\mathbf{H}_G^*(\mathcal{Q}'(M))$ when G is nonabelian is out of reach because it is not known if $\mathbf{H}_G^*(\mathbf{C})$ is a finitely generated vertex algebra.

8. Concluding Remarks and Open Questions

Our descriptions of $\mathbf{H}_G^*(\mathcal{Q}'(M))$ and $\mathbf{H}_G^*(\mathcal{Q}(M))$ are given relative to the family of vertex algebras $\mathbf{H}_K^*(\mathbf{C})$ for various connected normal subgroups $K \subset G$. An important open question in this theory is to describe $\mathbf{H}_G^*(\mathbf{C})$ for any G . Note that for $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, we have $\mathbf{H}_G^*(\mathbf{C}) = \mathbf{H}_{G_1}^*(\mathbf{C}) \otimes \cdots \otimes \mathbf{H}_{G_n}^*(\mathbf{C})$, where G_1, \dots, G_n are compact, connected Lie groups with Lie algebras $\mathfrak{g}_1, \dots, \mathfrak{g}_n$, respectively. We already know how to describe $\mathbf{H}_G^*(\mathbf{C})$ when G is abelian, so it suffices to assume that G is simple.

Question 8.1. *Recall from [12] that for simple G , the weight one subspace $\mathbf{H}_G^*(\mathbf{C})[1]$ is isomorphic to $\text{Hom}_G(\mathfrak{g}, S(\mathfrak{g}^*))$, which is a finitely generated, free module over $\mathbf{H}_G^*(\mathbf{C})[0] =$*

$S(\mathfrak{g}^*)^G$, by a theorem of Kostant. Is there a similar representation-theoretic description of $\mathbf{H}_G^*(\mathbf{C})[n]$ for any n ?

Question 8.2. *Is $\mathbf{H}_G^*(\mathbf{C})$ finitely generated as a vertex algebra? Can we find a set of generators?*

Question 8.3. *Can we compute the character $\chi(G) = \sum_{p,n} \dim \mathbf{H}_G^p(\mathbf{C})[n] z^p q^n$? Does $\chi(G)$ have any nice properties (modularity, relations to other objects from classical Lie theory, etc.)? For compact M , how is $\chi(G, M) = \sum_{p,n} \dim \mathbf{H}_G^p(\mathcal{Q}'(M))[n] z^p q^n$ related to other known invariants of M ?*

Question 8.4. *Is there a vertex algebra valued equivariant cohomology $\mathbf{H}_G^*(M)$ for any topological G -space M which coincides with $\mathbf{H}_G^*(\mathcal{Q}'(M))$ when M is a smooth G -manifold? In particular, we must have $\mathbf{H}_G^*(pt) = \mathbf{H}_G^*(\mathbf{C})$; what is the topological interpretation of $\mathbf{H}_G^*(\mathbf{C})$?*

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